THE

MATHEMATICAL GAZETTE

EDITED BY

W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF

F. S. MACAULAY, M.A., D.Sc.

AND

Prof. E. T. WHITTAKER, M.A., F.R.S.

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The Mathematical Association.

THE ANNUAL MEETING of the Mathematical Association will be held at the LONDON DAY TRAINING COLLEGE, Southampton Row, London, W.C. 1, on *Tuesday, January* 4, 1921, at 10 *a.m.* and 2.30 *p.m.*

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NOTICE.

The following Reports have been issued by the Association:—(i) "Revised Report on the Teaching of Elementary Algebra and Numerical Trigonometry" (1911), price 3d. net; (ii) "Report on the Correlation of Mathematical and Science Teaching," by a Joint Committee of the Mathematical Association and the Association of Public School Science Masters, price 6d. net; (iii) A General Mathematical Syllabus for Non-Specialists in Public Schools, price 2d. net. These reports may be obtained from the Publishers of the *Gazette*.

(iv) **Catalogue of current Mathematical Journals,** with the names of the Libraries where they may be found. Pp. 40. Price, 2s. 6d. net.

"Even a superficial study convinces the reader of the general completeness of the catalogue, and of the marvellous care and labour which have gone to its compilation."—*Science Progress*, Jan. 1916.

(v) **Report of the Girls' Schools Committee, 1916:** Elementary Mathematics in Girls' Schools. Pp. 26. 1s. net.

(vi) **Report on the Teaching of Mechanics, 1918** (*Mathematical Gazette*, No. 137). 1s. 6d. net.

(vii) **Report on the Teaching of Mathematics in Public and Secondary Schools, 1919** (*Mathematical Gazette*, No. 143). 2s. net.

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VOL. X. DECEMBER, 1920. No. 149.

THE DURHAM SUMMER COURSE IN MATHEMATICS FOR TEACHERS IN SECONDARY SCHOOLS.

BY G. J. B. WESTCOTT, M.A.

TOGETHER with our fellow-students of the Courses in French, Geography and History, we assembled on the evening of the 30th July at the University Lecture Rooms, Palace Green, Durham, and were welcomed on behalf of the University by the Dean of Durham and Canon Cruickshanks, who promised us that everything should be done by the authorities to make our visit to their historic city a pleasant one, a promise which it is needless to say was amply fulfilled.

The next morning the section in Mathematics, to the number of about thirty-five, met the Lecturer and Organiser of the Course, Prof. W. P. Milne of Leeds University, who explained that the course would include:

(1) Lectures and discussions on the teaching of School Mathematics.
(2) Lectures on the principles and practice of some modern methods of computation.
(3) Lectures on the lives of some of the great Mathematicians and Physicists.

The teachers attending the course came from all parts of England and Wales, and were representative of the Public Schools, the Grammar Schools, the County Secondary Schools, the City Secondary Schools, Pupil Teacher Centres, and Girls' High Schools. Prof. Milne began his Lectures on the teaching of School Mathematics by pointing out that we were aiming at the best methods of educating our pupils in Mathematics, so as to make the subject of real and living interest to them, and that while the problems of time-tables and examinations would be always with us and were matters of great importance, yet for each of us circumstances would differ, and we must adapt the ideal which we evolved each in our own particular way. In this manner we eliminated from our discussions what might be styled school politics.

The Lecturer first dealt with Geometry. The subject arose as practical land surveying among the Egyptians, but their successors, the Greeks, made it highly abstract, basing it on a minimum number of axioms. Their text-book, Euclid, had come to our schools through our universities, and we had attempted until very recently to teach young pupils Geometry as a philosophical subject, and had of course met with great difficulties.

One of these difficulties had been Euclid's Parallel Axiom, and it had been shown by the work of Gauss, W. and J. Bolyai, and N. Lobatschewsky, that consistent geometries could be built up in which Euclid's Axiom did not hold. Accounts of such systems could be found in books like R. Bonola's *Non-Euclidean Geometry*. Again, there was the subject of incommensurable quantities, introduced as a consequence of the theorem of Pythagoras, of which many other examples have arisen in the later developments of Mathematics. This subject had only in recent years been placed upon a rigid basis by the labours of Dedekind, Mayer, Cantor and Cauchy. It was the great glory of the Greeks that they had in Euclid given a sound test for the proportionality of such quantities, and had defined equality by an infinite number of inequalities ; but the argument was much too difficult for the average schoolboy. Frankly then, the subject should not be treated philosophically for the beginner, involving as it frequently does the proof of what to him is obvious.

There should be a course of Practical Geometry, in which intuition takes a prominent place, followed by a course in Computative Geometry with the computation stressed. Later there should be a course in Formal Geometry.

Practical Geometry would include the use of instruments, the drawing of right angles, the bisection of lines and angles, the drawing of parallels defined as lines always the same distance apart, and the plotting of isolated points on a locus to suggest the locus.

The area of a triangle should be obtained as in Fig. 1.

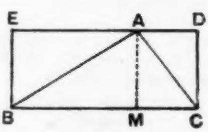

FIG. 1.

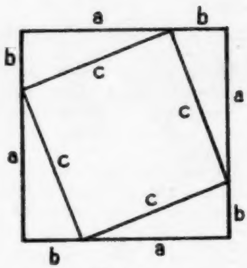

FIG. 2.

The theorem of Pythagoras might be proved as indicated in Fig. 2.

$$(a+b)^2 = 4 \times \tfrac{1}{2}ab + c^2,$$

and therefore $$a^2 + b^2 = c^2.$$

Full use of congruence should be made, preceded by reasons. No full-dress proof is required.

Similarity should be treated early as map-drawing to scale, and it should be pointed out that pairs of corresponding lines are proportional, and pairs of corresponding angles are equal.

Simple illustrations should be given showing that if two straight lines are in the ratio say of $3:1$, then the areas of the squares on them are as $9:1$, and the volumes of the cubes having them as edges are as $27:1$.

Circles should be treated as similar figures, and thus if we have a number of circles of diameters $d_1, d_2, d_3, \ldots$, whose circumferences are $p_1, p_2, p_3, \ldots$, and whose areas are $A_1, A_2, A_3, \ldots$, then

$$\frac{p_1}{d_1} = \frac{p_2}{d_2} = \frac{p_3}{d_3} = \ldots = \text{constant},$$

and $$\frac{A_1}{d_1^2} = \frac{A_2}{d_2^2} = \frac{A_3}{d_3^2} = \ldots = \text{constant}.$$

A discussion followed in which it was pointed out that, by means of paper folding, convincing reasons could be given for many important propositions, *e.g.* "equality of the base angles of an isosceles triangle," and "sum of the angles of any triangle is equal to two right angles." It was also suggested that the construction of triangles with given elements and the use of scissors might be the method of dealing with congruence; that dissection of figures might be used to prove equality of areas; and that, generally, the whole of the Practical Geometry Course of the Lecturer lent itself to treatment in a Mathematical Laboratory.

In a subsequent lecture Prof. Milne suggested that Computative Geometry should be introduced by the definition of the sin, tan, etc., of angles in the first quadrant, and the calculation of an example or two, *e.g.* sin 37°, tan 54°. Use of tables and their application to simple problems in heights and distances would follow, all triangles not right-angled being cut into two right-angled triangles by dropping perpendiculars. In solid geometry, propositions such as "two perpendiculars to a plane are parallel" and its converse ought to be taken as intuitive. A type of question that deserved considerable attention was the extension of known theorems on triangles and circles to the corresponding theorems on tetrahedra and spheres. Another useful type of question is the following:

"How many lines can be drawn to meet two lines p and q, and to be parallel to a third line r, no two of the lines p, q, r being coplanar?"

Problems due to the dissection of solid figures such as are illustrated by the annexed diagrams are very instructive:

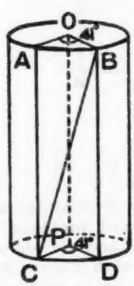

FIG. 3.

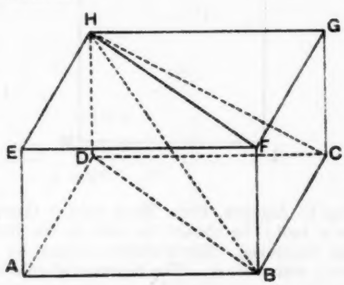

FIG. 4.

E.g. in Fig. 3, find the length of BC and the inclination of BC to the horizontal.

And in Fig. 4, find the length of BH, the angle between the edges of the figure and a diagonal plane, and the angle between the planes HBC and $ABCD$.

The formal presentation of Geometry should follow on the ordinary lines, and from this point ground could be rapidly covered by the pupil.

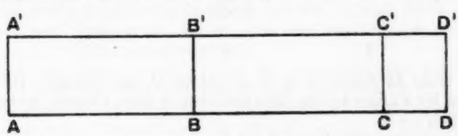

FIG. 5.

In the formal presentation of Ratio it should be taken as intuitive that rectangles such as $ABB'A'$, $CDD'C'$, having $ABCD$ and $A'B'C'D'$ straight

lines, have their areas proportional to their bases AB, CD, whether these are commensurable or incommensurable.

The very important proposition "The areas of similar triangles are proportional to the squares on corresponding sides" should then be proved as follows:

$$\frac{ABKH}{ALMB}=\frac{HA}{AL}=\frac{HA}{AB},$$

$$\frac{A'B'K'H'}{A'L'M'B'}=\frac{H'A'}{A'L'}=\frac{H'A'}{A'B'}.$$

But $$\frac{HA}{AB}=\frac{H'A'}{A'B'} \text{ by similarity of the triangles;}$$

$$\therefore\ \frac{\Delta ABC}{\Delta A'B'C'}=\frac{\text{rectangle } ABKH}{\text{rectangle } A'B'K'H'}=\frac{\text{square } ALMB}{\text{square } A'L'M'B'}.$$

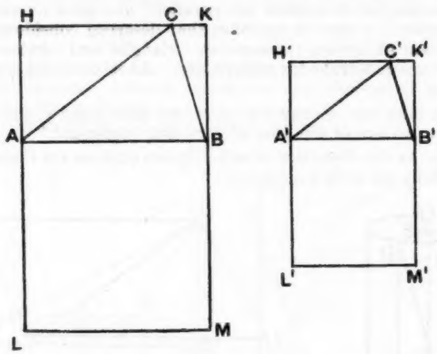

FIG. 6.

Coming to Algebra, Prof. Milne said it should be acquired by the average pupil as a tool; he should be able to use it to deal with problems and to build up formulae. Manipulation comes by practice, and we should not consciously strive for it. The beginner should deal first with positive numbers only, learning to use formulae, to plot statistics and to make interpolations, to construct very easy formulae and to solve simple equations. Directed numbers should then be introduced by means of debts, temperatures, distances west, time before noon, etc. Addition and subtraction of directed numbers give no trouble, but with respect to multiplication and division a difference of opinion arises. Some teachers incline to take the rule of signs as axiomatic, others give illustrations. There are two common methods of illustration, the "train" method and the "capstan" method.

Noon

W $\qquad\qquad$ O $\qquad\qquad$ E

Suppose a train at Noon to be at a station O, and suppose its speed to be 40 m/h. Now let s miles be its distance from O after t hours, when its velocity is v m/h. Then $s=vt$.

Now if the train is going from W to E,

where is it at 5 p.m.? $\quad v=+40,\quad t=+5,\quad s=+200;$

where is it at 10 a.m.? $\quad v=+40,\quad t=-2,\quad s=-80.$

Again, suppose the train is going from E to W,

where is it at 3 p.m. ? $\quad v=-40, \quad t=+3, \quad s=-120;$

where is it at 8 a.m. ? $\quad v=-40, \quad t=-4, \quad s=+160.$

In the "capstan" method a sailor pushes with a force of 50 lb.-weight at the end of an arm 3 feet long. The turning effect is easily seen to be measured by the product of the two numbers representing the size of the force and the length of the arm. A counter-clockwise rotation is as usual considered to be positive, a clockwise rotation is then negative. If the capstan bar points E it will be considered positive, and if W negative. If the sailor pushes N the force will be positive, and if he pushes S negative.

Four cases then arise as in the diagrams :

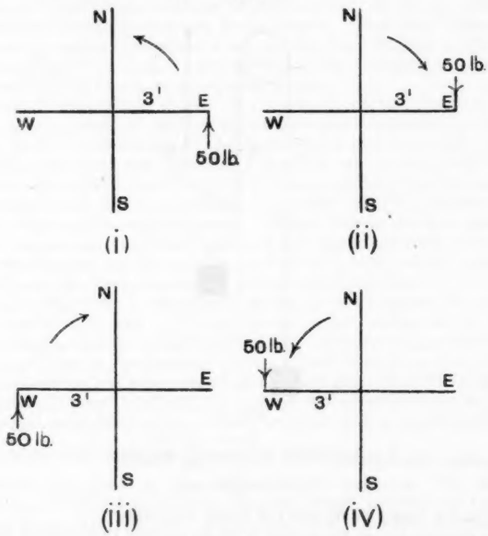

FIG. 7.

In (i) $T=(+50)\times(+3)=+150.$

(ii) $T=(-50)\times(+3)=-150.$

(iii) $T=(+50)\times(-3)=-150.$

(iv) $T=(-50)\times(-3)=+150.$

The rule for multiplication having been developed, that for division immediately follows, and thence the use of formulae and manipulative drill with polynomials involving directed numbers.

A great deal of interest was taken in the discussion that followed, showing that a good many teachers had found real difficulty on this point. Most agreed that some illustration was desirable. Several suggested that the minus sign should be regarded as a vector which turned the direction through two right angles. The *Gazette* gives at length Mr. S. Lister's suggestions as to filling up a multiplication diagram by addition and subtraction.*

* *v.* p. 173.

The Lecturer then continued his suggestions on the teaching of Algebra. Simple equations and then simultaneous equations should be solved in the ordinary way, and should be followed by problems. Answers to equations in two unknowns should always be shown thus:

x	3
y	5

x	-4	7
y	9	-6

to bring out the connection between x and y.

Further examples of the construction of easy formulae should follow, such as, Cost of telegram, C pence, in terms of number of words n,

$$C = 9 + \tfrac{1}{2}(n - 12).$$

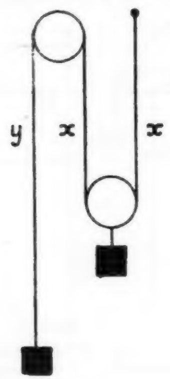

FIG. 8.

String running round pulleys and supporting weights,

$$y + 2x = 24 \text{ (say)},$$

the upper pulley being fixed and the lower movable.

Obsolete income-tax at 9*d*. in the £ on an income in excess of £150,

$$\frac{T}{I - 150} = \frac{3}{80}.$$

Then drill upon quadratic expressions, *e.g.* $(5x - 4)(2x + 3)$, multiplying at sight and refactoring. Following upon this again should come solution of quadratics by factors. Equations such as $3u - 5t = 7$ should be taken, and u expressed in terms of t, and t expressed in terms of u.

Being given a set of corresponding values of x and y such as the annexed, by plotting on squared paper to find a connection between x and y.

x	1	2	3	4	5	6	7
y	5	8	11	14	17	20	23

In the progressions the examples usually given are very artificial and lacking in interest. A good example of an Arithmetical Progression is the

winding of cloth upon a drum, and of a Geometrical Progression the calculation of the value of annuities.

The teaching of the Binomial Theorem can be made instructive by showing how Newton made use of intuition in its discovery. He multiplied out $(1+a)(1+b)(1+c)\ldots$, and then put $x=a=b=c=\ldots$, thus establishing the Theorem for a positive integral exponent. He then assumed the law to be true for other exponents, and verified for the cases $(1+x)^{\frac{1}{2}}$ by taking a square root in the usual way, $(1+x)^{\frac{1}{3}}$ by putting $(1+x)=(a_0+a_1x+a_2x^2+\ldots)^3$ and comparing coefficients, and for $(1+x)^{-1}$ by a long division.

Following the Lectures on Algebra a discussion took place on the teaching of logarithms. The general opinion was that they should be introduced into the Arithmetic Syllabus as soon as the pupils understand the square-root rule and the logs. of sufficient numbers between 1 and 10 can be calculated by Briggs's method to enable a graph to be drawn. The pupils then grasp the idea that every number possesses a log. to the base 10, and a set of Tables, which it is explained to them is calculated in a less laborious way, can then be safely placed in their hands as a calculating tool.

The writer of this article showed the members of the course a method of drawing a good graph of $\log_{10}x$, which was communicated to him by Mr. A. Dufton, H.M.I. of Secondary Schools in Devon and Cornwall. It depends upon the principle that the ratio $\log_a x : \log_b x$ is constant for all values of x. The pupils can be convinced of this by plotting carefully on the same squared paper $y=\log_2 x$ and $y=\log_3 x$, which are easily drawn between $x=1$ and $x=16$ and with considerable accuracy. When this is realised, another piece of squared paper may be taken and $y=\log_2 x$ again carefully plotted. Now, if on the same paper we plot $y=\log_{10}x$, but on a scale whose y coordinate is 5 times as large, we have at once the points corresponding to $x=1$ and $x=10$, and we notice that the y coordinate of $\log_{10}x$ is $1\frac{1}{2}$ times the y coordinate of $\log_2 x$ when $x=10$, and therefore for all other values of x. Thus each point on $y=\log_{10}x$ can be obtained from the corresponding point on $y=\log_2 x$ and $y=\log_{10}x$ can be accurately drawn. Incidentally a good deal of the theory of logarithms has been hinted at and can later on in the pupil's algebra course be pressed home, and still later, when the Calculus is being dealt with, the differential coefficient of $\log_{10}x$ may be read off as a gradient of the curve, and the fundamental equation $\frac{dy}{dx}=\frac{0{\cdot}43}{x}$ obtained.

Prof. Milne then took up the subject of the Calculus. He showed that, by using a graph the very difficult idea of a limit, which is so apt to puzzle the average boy, was rendered unnecessary for him, and he had the great advantage of being able to use the Calculus as a tool; while at the same time, the mathematical specialist obtained an acquaintance with the Calculus earlier than he would otherwise have done, and yet learned nothing that he would have subsequently to unlearn. By dealing first with simple algebraic expressions, the pupil can learn to apply the Calculus to examples in Geometry, Algebra, Mechanics and Engineering without being distracted by the calculation of the differential coefficients of elaborate functions, and then later he may extend his knowledge to the more difficult cases of the trigonometrical functions and logarithms.

Having learnt to use the Calculus at an earlier age than was usual in the past, he can apply it in his experimental work in Science, to the mutual advantage of his Science and his Mathematics. The sum total of his work will be lightened because he will be able to jettison all the calculus-dodging devices which have crept into the teaching of Physics.

As the result of a discussion on experimental work in connection with Mathematics, the members of the course together with Prof. Milne asked the writer, who controls both the Mathematics and Physics of his own school, to read a paper on the subject. While holding that the ideal arrangement

is the closest co-ordination between the teaching of Mathematics and Physics, the writer admitted that circumstances might arise in which a school or department of a school had to organise experimental work in what are usually styled mathematical subjects in a mathematical laboratory. He proceeded to outline a course to include the teaching of Geometry, Mass, Density, Time, Statics and Dynamics, and made suggestions as to the equipment of a room for the purpose. He made a special point of the great educational value to the pupil of discovering laws connecting two variables, and not merely verifying them, by plotting observations on squared paper. He showed how pupils who had learnt to use logarithms could establish formulae of the type $x^m y^n = \text{constant}$, where m and n may have like or unlike signs by plotting $\log y$ against $\log x$, and so obtaining a straight line graph whose gradient gave the ratio of $-m$ to n. Great interest was taken in the discussion that followed. Several teachers pointed out that a laboratory on the lines suggested could be made to co-ordinate with Geographical teaching, and others suggested further apparatus to increase its efficiency. Prof. Milne said they were hoping to develop mathematical laboratories on similar lines at Leeds, and he thought the problem would also arise in continuation and central schools.

In the Lectures on Computation, Prof. Milne showed us, as examples of iterative processes, the solution of the equation $x = 70 \sin x$, the obtaining of a table of cube roots from a table of square roots, and the solution of a numerical cubic equation arising in practice. He explained to us Graeffe's method of solving numerical equations by dispersion of the roots through repeated squaring. This method, though invented so long ago as 1830, was only brought into prominence through the War, it being much used in the solution of the standard equation of the 8th degree, upon which depends the stability of an aeroplane. Prof. Milne also worked out at length the method of construction of a 7-figure table of Logarithms by Briggs's method, commenting that English courses of mathematics too often failed to give any idea of a sense of number.

Lastly, the Lecturer gave us a clear exposition of Nomography, and pointed out the great use of nomograms in industry, giving instances of time and money saved by their application to problems arising in the industries of Leeds. At the request of one of the ladies, he showed how a nomogram might be constructed to enable marks to be easily scaled, and this appealed to most of those present.

In the Biographical Lectures, Prof. Milne said all schoolboys ought to have some idea of the lives and personalities of the great mathematicians and scientists, and to realise that they were not all of the type of the savant who is represented on the poster advertising this year's meeting of the British Association at Cardiff. At any rate, Carnot, Maclaurin, Kelvin and Rayleigh, to mention only a few, were notable exceptions. The Lecturer proceeded to point out how certain branches of mathematics came into prominence at particular times, instancing that while at the present time Analysis and Function Theory held the field, forty years ago it was held by the Invariant school of Cayley, Sylvester and Salmon, details of whose careers were given. Then followed an account of Babbage and his calculating machine, and it was noted that the celebrated engineer, Joseph Whitworth, was at one time a mechanic engaged in its construction. After this came the life of another great engineer, Rankine, who was the cousin of Graham, the discoverer of the Laws of Diffusion.

Graphs and their inventors were then considered. A life of Descartes was read, and it was shown how slowly his ideas were taken up in England, so that when they were used by Newton he had to recast his results into the forms of pure geometry for purposes of publication. The advantage of the method is that rapid progress may be made in a new subject and much ground covered, and then later a return may be made to give rigid proofs and clear up detail.

A short account of the life of Prof. Perry and his work was given, and then Nomography and Maurice d'Ocagne, Argand and his Diagram, and the work of Abel, Riemann and Neumann were briefly mentioned. It was pointed out that the Riemann surfaces came in as a side issue when Riemann was investigating Trigonometrical series. Monge, the inventor of Descriptive Geometry, discovered a geometrical method which dispensed with the mathematical expert, and this was also the result of the invention of Graphical Statics, whereby all the calculations of bridgework could be carried out by a draughtsman.

A life of Clerk Maxwell was read, and his researches in Optics, Electricity and Magnetism, the Kinetic theory of Gases, and the motion of Saturn's Rings mentioned; but probably the most profound impression made on us was when the Lecturer, in his inimitable manner, quoted the remark made to Clerk Maxwell by his aunt at the breakfast table, when as a lad in an absent-minded mood he neglected his porridge, "James, you are in a prop!"

The career of Todhunter, the text-book writer, was next traced, and his great characteristic, reverence for authority, was pointed out.

Then followed a detailed life of Sophie Kovalevski, pupil of Weierstrass, colleague of Mittag-Leffler, intimate friend of Dostoievski, the Russian novelist, greatest of women mathematicians, and pioneer of the movement which opened the universities to women.

The Lecturer closed with the life of Faraday. Sprung from a Yorkshire moorland stock, son of a blacksmith, errand-boy to a bookseller, to whom he was later apprenticed as a bookbinder, he became Davy's laboratory assistant at the Royal Institution. With Davy he spent two years in a tour through Europe, visiting Paris, Genoa, Rome and Geneva, and meeting many of the leading continental scientists, and then returned to make his great discoveries in Chemistry and Electricity and Magnetism, which brought him fame and honour. His sturdy independence, his deep religious feeling, his love of children were all referred to, and his character summed up in Tyndall's phrase, "Just a faithful Knight of God."

During the Course social activities were not neglected. At the beginning of the Lectures four members were elected to organise this side of our life, and to these ladies and gentlemen everyone of us is largely indebted for the very pleasant fortnight spent in Durham. Boating and tennis and sing-songs were arranged, and excursions made to places of interest in the neighbourhood, including the Cathedral, Durham Castle, Finchale Priory and Brancepeth Castle. A visit was paid to a neighbouring coal-mine, and a tour was made of Armstrong-Whitworth's works at Newcastle. After the latter visit a merry party, including Prof. Milne, dined together and went on to the Newcastle Theatre. The greatest good-feeling reigned among all members of the Course, and we were all unfeignedly sorry when a most interesting and enjoyable meeting came to its close.

G. J. B. WESTCOTT.

GLEANINGS FAR AND NEAR.

59. In talking to men whose brows were smeared with dirt, and whose hands were black with soot, I found upon them the marks of intellectual minds, and the proofs of high character; and I conversed with men who, in their way, and in many ways bearing upon the purposes of life, were far my superiors . . . The feeling prevailing in my mind was that of the intellectual capacity manifest in the humbler orders of the population of Manchester . . . While the institutions and customs of man set up a barrier, and draw a great and harsh line between man and man, the hand of the Almighty stamps his finest impress upon the soul of many a man who never rises beyond the ranks of comparative poverty and obscurity. . . .—*Life of Sedgwick*, ii. p. 46.

VECTOR ANALYSIS IN A UNIVERSITY COURSE.

By C. E. Weatherburn, M.A., D.Sc.

Since due recognition is now being more and more widely given to the importance of Vector Analysis for three-dimensional work in mechanics, geometry and mathematical physics, the time is opportune to consider the place which that subject merits in a University degree course. A student who undertakes anything like research work in applied subjects is considerably handicapped if he has had no training in Vector Analysis; but the need for it is felt much earlier than the stage at which research is generally begun. Every University course contains subjects which are less useful than Vector Analysis; and on the ground of utility alone there is no question that the latter deserves a place in the curriculum. It will, however, be found quite unnecessary to displace anything else, because the time occupied in teaching the essential parts of vector algebra and calculus will be saved, even during a three years' course, by the application of this method to the three-dimensional parts of mechanics and mathematical physics.

It is, perhaps, not generally recognised that the subject of Vector Analysis belongs essentially to the domain of *pure* mathematics. The fundamental length-vector is a geometrical quantity whose magnitude is a length; and vector algebra involves nothing but elementary geometry, algebra and trigonometry. Certainly its most important applications are in mechanics and physics, but the same is true of certain other branches of pure mathematics. The differential and integral calculus of vectors involves only geometry and the algebraic calculus. In the analysis itself there is no need to make mention of a single physical quantity. The theory of the linear vector function, whether it employs dyadics, tensors or matrices, is purely algebraic, and in no way dependent upon mixed mathematics, though it has extremely important applications in elasticity and the electromagnetic theory.

The time for teaching the different parts of the subject will depend to some extent upon local conditions. Generally, however, it will be found advisable to give a short course in *vector algebra* to "honour" students in their first year, and to "pass" students either at the end of the first or at the beginning of the second year. As remarked above, this part of the subject is quite elementary, and may be disposed of in six or seven lectures. These would treat of sums of vectors, the various products of two or three vectors, and some geometrical applications. A little later, when the student possesses a fair knowledge of the algebraic calculus, differentiation of vectors with respect to a single variable may be treated. In elementary applications the single variable will usually be the time variable t or the arc-length s.

Even this elementary knowledge of Vector Analysis will be found exceedingly useful in mechanics, and to some extent also in solid geometry. By means of it practically all the important principles of *mechanics* are just as easily proved for three dimensions as for two, and a great simplification is introduced into the work. The vector moment or torque about the origin, of a force* $\mathbf{F}$ localised in a line through the point whose position vector is $\mathbf{r}$, is given by the vector product $\mathbf{r}\times\mathbf{F}$; and the (scalar) moment about any axis through the origin is the resolved part of the vector moment in that direction. Similarly the moment of any other localised vector may be defined. The moment of the velocity of a particle is represented by $\mathbf{r}\times\mathbf{v}$, which is twice the areal velocity about the same point. The angular momentum or moment of momentum of the particle is $\mathbf{r}\times(m\mathbf{v})$; and the A.M. about any axis through the origin is the resolved part of this vector in the direction of the axis. The work of a force $\mathbf{F}$, during a displacement $\mathbf{d}$ of the particle acted upon, is given

* In printing, vectors are usually denoted by Clarendon symbols. For manuscript and blackboard work, Greek letters and script capitals will be found convenient.

by the scalar product $\mathbf{F}\cdot\mathbf{d}$; and it is merely a corollary that the work of the resultant of several forces is the sum of the works of the components. The activity of a force $\mathbf{F}$ at any instant is $\mathbf{F}\cdot\mathbf{v}$, where $\mathbf{v}$ is the velocity of the particle acted upon.

The relative position of one point P with respect to another, O, is determined by the vector $\mathbf{r}=\overrightarrow{OP}$. The velocity of P relative to O is the rate of change of its relative position, and is therefore given by the vector $\mathbf{v}=\frac{d\mathbf{r}}{dt}$. The relative acceleration is similarly the rate of increase of the relative velocity, so that

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d^2\mathbf{r}}{dt^2}.$$

The theorems of the vector addition of velocities or accelerations are an immediate consequence. In the case of a system of particles the importance of the centre of mass, and the theorems connecting the velocity and acceleration of this point with the momenta of the separate particles and the vector sum of the forces acting on the system, follow directly from the formula

$$\bar{\mathbf{r}}=\frac{\Sigma m\mathbf{r}}{\Sigma m},$$

which defines the centre of mass. The formulae associated with rotating axes can be proved vectorially almost in one line; and the single vector formula is much easier to remember than the triad of scalar ones. In three-dimensional Statics the equations of equilibrium for a rigid body are reduced to two; and all the spatial work, such as Poinsot's reduction of a system of forces on a rigid body, is wonderfully abbreviated and simplified.

In *solid geometry* the essential things are vector quantities. Each point is specified by its position vector relative to an assigned origin, and each element of a surface by its position vector and vector area. The equation of a plane takes the simple form $\mathbf{r}\cdot\mathbf{n}=p$, where $\mathbf{n}$ is a unit vector perpendicular to the plane, and $\mathbf{r}$ the position vector of a current point on the plane. The whole geometry of the plane may be deduced very concisely from this equation. The vector equation of the straight line through the point $\mathbf{a}$ parallel to the vector $\mathbf{b}$ is $\mathbf{r}=\mathbf{a}+t\mathbf{b}$; and the geometry of non-intersecting lines involves nothing beyond the scalar triple product of vectors, corresponding to the determinant of 3×3 elements in coordinate geometry. The coordinate equivalent of any vectorial result is a mere corollary, whether the axes are rectangular or oblique. Thus the heavy, artificial and lengthy argument in the case of oblique axes becomes unnecessary; for it is as easy to expand our formulae in terms of any three non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as in terms of the mutually perpendicular vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. The standard equation of a sphere may be written $\mathbf{r}^2-2\mathbf{r}\cdot\mathbf{c}+k=0$, from which the geometry of the sphere may be neatly and concisely deduced. Finally the curvature and torsion of curves are very easily investigated in terms of the derivatives of the position vector of a current point on the curve, with respect to the arc-length s. The method is perfectly elementary, and the results appear in a form which is easy to remember.

The *advanced part* of Vector Analysis naturally begins with the differential operations which yield the gradient of a scalar point-function, and the divergence and curl of a vector point-function. We are confronted with these as soon as we enter upon higher applied mathematics; and the student who has to do so without the vectorial equipment finds himself seriously handicapped. There is here, after all, only a very little to learn. All that is really important about the differential operations would make only one chapter of reasonable size; while another of the same size could contain the theorems on line-, surface-, and volume-integrals which are continually employed in advanced mathematics. These are Gauss's divergence theorem, Stokes's theorem, and a few others which are immediately deducible from them. Such

matters belong to the domain of pure mathematics; and their teaching should not be consigned to lectures or books in mixed mathematics. The student is thereby apt to gain the impression that the theorem is part of the applied subject under discussion. Lectures on advanced calculus are their appropriate setting; and six or seven lectures could supply all the necessary information. An "honour" student will probably need these at the end of his second year, in readiness for the higher applied work he is likely to have in his third. "Pass" students may not reach a standard, during their degree course, at which this work will be necessary. But that will depend upon the nature of the course.

Linear vector functions are hardly met till this stage; but their treatment is quite elementary and almost entirely algebraic. The theory may be developed in terms either of dyadics or of tensors, which belong to the province of multiple algebra. The resolved form of a dyadic is really equivalent to a matrix, and the theory of dyadics is parallel with that of matrices. But the dyadic has certain advantages, among others that it need not be expressed in resolved form, but in a shorter form as the sum of three dyads. Linear vector functions are of constant occurrence in elasticity and the electromagnetic theory; and the student will experience far fewer difficulties in these subjects, if he has had an earlier and separate introduction to the linear function. Without such a preparation he must find the theory of elasticity very heavy. The variable dyadic is a great help in the treatment of heterogeneous strain. The geometry of central quadric surfaces is also rendered more compact by the use of dyadics. So also is the treatment of moments and products of inertia, and the motion of a rigid body about a fixed point.

It is a mistake for the student to leave the consideration of vector methods till he has finished his course. Not only does he lose much before this stage, but he will be all the longer in acquiring that familiarity with these methods which is necessary for their effective employment, and which comes only with time and practice. Macaulay has remarked that "no noble work of imagination was ever composed by any man, except in a dialect which he had learned without remembering how or when." The two cases are hardly parallel; but we may safely say that the earlier a student begins to learn the vectorial mode of thought and expression, the greater will be his facility in three-dimensional calculations, and the better will be the quality of his later work in this direction.

C. E. WEATHERBURN.

Ormond College, University of Melbourne,
August, 1920.

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60. Fuller, of whom Queens' ought to be almost as proud as of Erasmus, has drawn the character of a learned and accomplished person: "Mathematics he moderately studieth to his great contentement, using it as ballast for his soul; yet to fix it, not to stall it, nor suffers it to be so unmannerly as to jostle out other arts." A mere mathematician, made up of unknown quantities, is a dreary and a melancholy spectacle—a tree without leaves.

And Coleridge is also reported to have said something to this effect: "I attribute very little efficacy to the demonstration of a proposition *per se*; but to the devotion and absorption of mind which the operation requires; not so much to the actual showing, upon paper, that 'Solid Parallelopipeds, which have the same altitude, are to one another as their bases,' as the few minutes required to prove it."—*Conversations at Cambridge* (Parker, 1836), pp. 12, 14.

61. Bread is better than paradise; because bread is better than nothing, and nothing is better than paradise—the confusion arises from both the "nothings" being used substantively; whereas it is only the first that is so used; the second is affirmative, and expresses "there is nothing better."—De Morgan, *Sophism*, *P.C.* xxii. p. 257.

MATHEMATICAL NOTES.

558. [A. 1.] *The Rule of Signs for a Product : the Completed Multiplication Table.*

In the course of a discussion, at the Durham Summer Course of Mathematical Teachers, on the Rules of Signs and methods of teaching them, I was rather surprised to find that the following method of obtaining the rules for multiplication of directed numbers was almost unknown to those present, and was quite unknown in its complete form. It does not seem to have appeared in print, and Professor Milne has suggested that I should send it to the *Gazette*.

The essence of the method is simply that the multiplication table is fundamentally built up by repeated addition, the multiplicand being added at each step to obtain the successive products, thus:

$$2 \times 1 = 2, \quad 2 \times 2 = 2 + 2 = 4, \quad 2 \times 3 = 4 + 2 = 6, \text{ etc.}$$

Conversely, of course, if any product is given those that have gone before can be obtained by subtraction; thus, if

$$13 \times 12 = 156, \quad 13 \times 11 = 156 - 13 = 143, \text{ etc.}$$

The multiplication table can therefore be written in the following compact form, each product being obtained as above:

	0	1	2	3	4	×
	0	4	8	12	16	4
	0	3	6	9	12	3
	0	2	4	6	8	2
	0	1	2	3	4	1
	0	0	0	0	0	0

The commutative law is strikingly displayed in this arrangement, and the figures at the top and the side may be considered as interchangeable from the point of view of multiplier and multiplicand.

A class which has been dealing with the addition and subtraction of directed quantities will readily suggest the idea of extending the table to include negative factors, and the following is at once obtained by repeated subtraction:

	−4	−3	−2	−1	0	1	2	3	4	×
	−16	−12	−8	−4	0	4	8	12	16	4
	−12	−9	−6	−3	0	3	6	9	12	3
	−8	−6	−4	−2	0	2	4	6	8	2
	−4	−3	−2	−1	0	1	2	3	4	1
	0	0	0	0	0	0	0	0	0	0
					0	−1	−2	−3	−4	−1
					0	−2	−4	−6	−8	−2
					0	−3	−6	−9	−12	−3
					0	−4	−8	−12	−16	−4

The filling in (by means of addition) of the portion now left vacant in the above leads to positive products. The consistency of the results in this (third) quadrant of the table is again a striking tribute to (or verification of) the commutative law, as the same result is obtained by proceeding to the third quadrant from *either* the second or fourth quadrants.

A skeleton of the table can now be made to generalise the results :

– ve number	0	+ ve number	×
Product – ve	0 0 0 0	Product + ve	+ ve number
0 0 0 0	0	0 0 0 0	0
Product + ve	0 0 0 0	Product – ve	– ve number

Alternatively, the original table may be used by writing the zeros and the generalised results in coloured chalks.

It is interesting to compare the table with the results of plotting graphically $s=vt$, where s, v, t are directed numbers with assigned zeros, as suggested by Professor T. P. Nunn in his *Teaching of Algebra and Trigonometry* (Longmans). It condenses the results of that method into a tabular form. The one method, however, appears to emphasise the *inevitable form* of the result while disguising the *vector nature* of the quantities, while the other emphasises this *vector nature* of the quantities and disguises the generality of the result.

The chief advantages of the method from a pedagogical point of view seem to be :

(1) That the fundamental idea of directed numbers as previously considered by the pupil, namely that of 'order,' is used.

(2) That the fundamental idea of multiplication, namely that of repeated addition, is used.

(3) That the idea of continuity plays an important part.

To write the multiplication table for positive factors in this compact form first occurred to the writer in preparing for publication *A First Book of Arithmetic* (Macmillan) some years ago.

I believe that at the present time we introduce our pupils to negative quantities much too early in the algebraic course, and in this way they lose the beauty of the whole conception. To my mind it would be much better to follow more closely the historical development in the construction of our curriculum and syllabuses, and to cover the content of much of the algebraic course before undertaking the formal treatment of negative quantities. In this way, the tremendous unifying and generalising power of the extension of number would make a powerful appeal to the pupil and prepare him for similar instances in his future work. Is it not the discovery of *the extension of the boundaries of number and similar concepts* which has constituted the genius of the work of most of the world's greatest mathematicians ? S. LISTER.

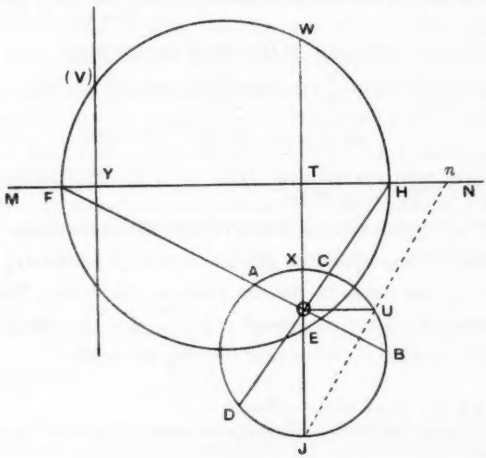
West Leeds High School.

559. [L[1]. 1. e.] *Conical Projection of a Conic.*

The laborious methods of many draughtsmen suggest that they are unaware that to obtain the conical projection of an original conic it is never necessary to *project* more than *three* properly selected points in the plane of the original

conic. If, while the original conic is either a circle, ellipse, parabola or hyperbola, the projection conic lies on either an ellipse, parabola or hyperbola, the number of points necessary to select and project will be the maximum, viz. three. It is proposed here to point out several ways of properly selecting three points in such cases, though for lack of room it will only be possible to particularise for cases in which the projection is an ellipse. The proposal thus restricted will be carried out if it is shown how to find on the original conic such points as project into the extremities of the principal (major and minor) axes of the projection ellipse.

The original conic $ABCD$ is supposed to be lying in the plane of the figure. Let MN be the line in which the plane that is parallel to the plane of projection, and that contains V (the vertex of projection), cuts the plane of the figure.



The line $Y(V)$ is supposed to cut MN at right angles in the point Y. Plane I. is the plane perpendicular to the plane of the figure, cutting it in the line $Y(V)$. The V belonging to this figure may be any point, out of the plane of the figure, on the circle that lies in Plane I., has Y as centre, and a radius equal to $Y(V)$.

Find the pole, O, of MN with respect to the original conic. Let the diameter through O cut MN at T, and cut the conic, between O and T, at X.

Case 1. T not identical with Y.

On the perpendicular to MN through T set off at equal distances on either side of MN two points, W and E, obtained thus. Through O draw a line parallel to MN and cutting conic at U. Then

if original conic is a circle take $TW = TU$.

" " " " parabola take $TW = OU$.

" " " either ellipse or hyperbola, let J be the opposite extremity to X of diameter through O. Join JU, cutting, or produce to cut, MN in n, and take $TW = Tn$.

Next describe a circle through E, W, and (V), cutting MN at F and H.

Through F draw a line through O cutting original conic at A and B.

" H " " " " " " " C and D.

Then A, B, C, D are the required points.

Proof. Denote projections of A, B, etc., by A', B', etc.

$$\text{Angle } F'O'H' = FVH = F(V)H = 90^\circ.$$

And, O being pole of MN, O' will be the centre of ellipse.

Lastly, OF and OH, on account of the selection of W and E, being conjugate w.r.t. original conic, $O'F'$ and $O'H'$ will be conjugate w.r.t. the ellipse.

Thus AB and CD project into principal axes.

Case 2. T identical with Y.

The construction only will be given here, the reader being asked to supply figure and proof for himself.

Through O draw a line parallel to MN, cutting original conic in a and b. If the original conic has a centre, produce the diam. TXO to cut conic again at d. a, b, X, and, in the case of a central original conic, d, are the required points.

HOWARD E. GIRDLESTONE.

560. [J. 2] (Cf. Note 540.) If this result were printed

$$x^6\left\{1+\binom{6}{1}(1-x)+\binom{7}{2}(1-x)^2+\binom{8}{3}(1-x)^3+\binom{9}{4}(1-x)^4 \right.$$
$$\left. +\binom{10}{5}x(1-x)^5+2\binom{10}{5}(1-x)^6x^2/(1-2x\overline{1-x})\right\},$$

it would agree with the solution given by J. K. Whittemore, *Annals of Mathematics*, Ser. II. ix. pp. 82-84.

G. N. BATES.

561. [R. 7. a.] A correction is needed in Note 553, as the change of velocity at the corner of the square or polygon is not $\frac{v^2}{r}$ (obviously) but $\frac{v^2}{r}\times t$, where t is the time taken to describe a side of the figure. This gives the average change of velocity per second to be $\frac{v^2}{r}$. I shall certainly make use of "Cymric's" method in future when teaching my pupils.

C. H. HARDINGHAM.

562. [D. 2. a. α.] *Convergence of Series.*

The usual tests for the convergence of series of positive terms may be stated thus:

If $\frac{u_{n+1}}{u_n}=A-\frac{B}{n}-\frac{u}{n^2}$, where A, B are constants, and u remains finite,* as n is indefinitely increased, the series Σu_n is convergent,

(1) If $A<1$, *or* (2) if $A=1$, $B>1$;

these being derived by comparison with the series

(1) $1+\alpha+\alpha^2+\ldots$, (2) $1^{-\beta}$, $2^{-\beta}$, $3^{-\beta}$, ...,

where $\alpha=\frac{A+1}{2}$, $\beta=\frac{B+1}{2}$ respectively. If $A=1$ and $B=1$, it is often said that higher tests are required; but, $|u|$ being finite $<C$, say, it can be rigorously proved that the series is divergent in the following simple manner. Consider the series whose terms are $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, ..., *preceded by any* $k+1$ *finite numbers*. For this series ultimately

$$\frac{v_{n+1}}{v_n}=\frac{n-k-1}{n-k}=1-\frac{1}{n}-\frac{k}{n(n-k)}<1-\frac{1}{n}-\frac{k}{n^2},$$

if $k=1$ at least.

* This is, of course, practically the same thing as $\frac{u_{n+1}}{u_n}=A-\frac{B}{n}+\frac{C}{n^2}+\ldots$, but without involving an infinite series in its statement.

If then $\frac{u_{n+1}}{u_n}=1-\frac{1}{n}-\frac{u}{n^2}$, where $|u|$ is always $<C$, we have only to take for k a number $>C$ to ensure that ultimately, *even if u is positive*, $\frac{u_{n+1}}{u_n}>\frac{v_{n+1}}{v_n}$. But Σv_n is divergent; $\therefore$ Σu_n is divergent. With this addition the rules cover all cases that ordinarily arise.

Two more points.

(1) It is not always remembered that (keeping still to series of positive terms) if $\frac{u_n}{v_n}$ is always finite, $<C$, say, and Σv_n is convergent, Σu_n is also convergent; for

$$|u_{n+1}+u_{n+2}+\ldots+u_{n+p}|<C(v_{n+1}+v_{n+2}+\ldots+v_{n+p})<C\epsilon<\epsilon',$$

where ϵ has the usual meaning. In this way we may often substitute for a series a much simpler series before applying the test involving $\frac{u_{n+1}}{u_n}$; or convergence may be obvious for the simplified series without any such test.

(2) In the comparison with the series $1^{-\beta}+2^{-\beta}+3^{-\beta}+\ldots(\beta+^{\text{ve}})$, where the ratio of the $\overline{n+1}^{\text{th}}$ term to the $n^{\text{th}}=\left(1+\frac{1}{n}\right)^{-\beta}$, it is unnecessary to use the binomial theorem for a fractional or negative index, *which is undesirable in a fundamental test which is required for the full discussion of the binomial theorem*, if (as Chrystal) we use the important and easily proved inequality $\frac{a^p-1}{p}>\frac{a^q-1}{q}$, if $p>q$ (a being any positive number). For it follows that

$$\frac{\left(1+\frac{1}{n}\right)^{-\beta}-1}{-\beta}<\frac{\left(1+\frac{1}{n}\right)^{+1}-1}{+1}\ (\beta \text{ being positive}),\ i.e. <\frac{1}{n};$$

and therefore $$\left(1+\frac{1}{n}\right)^{-\beta}>1-\frac{\beta}{n}>1-\frac{B}{n}-\frac{u}{n^2}$$

ultimately, if $B>\beta$ and u finite; so that the latter ratio $<$ that in the series $1^{-\beta}+2^{-\beta}+3^{-\beta}+\ldots$, which is convergent if $\beta>1$; which proves that Σu_n is convergent if $B>1$, β being taken as above.

If in any case the terms are not all positive, the corresponding tests become tests for the *absolute* convergence of the series in question.

July 19th, 1920. PERCY J. HEAWOOD.

563. [L[1]. 9 b.] *Note on the integration of the difference between two Fagnano arcs of an ellipse.* See *Elliptic Trammels and Fagnano Points*, by Prof. P. J. Harding. *Mathematical Gazette*, vol. vi. pp. 68-78; 117-124.

The object of the following paper is to give a simple proof of the property of the Fagnano integrals.

In the ordinary figure for the description of elliptic arcs by trammels $HK(a+b)$ and $hk(a-b)$, let I, i be the instantaneous centres; $\phi_1=HCI=hCi$ the eccentric angle of P and CF the perpn. on the normal IPi through P; ψ_1 the angle Ii makes with the major axis; ϕ_2 the angle HIi.

Since $$\cot\phi_2=\tan\psi_1=\frac{b}{a}\tan\phi_1,$$

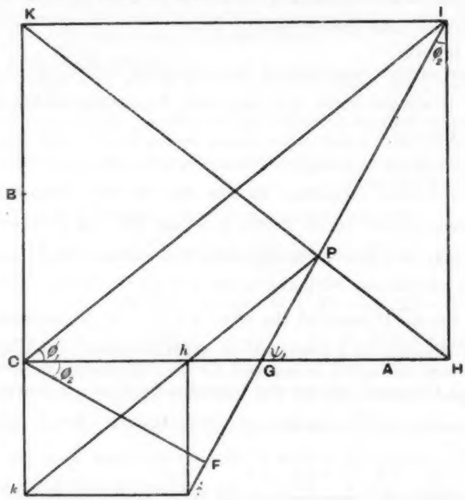
$$\therefore\ \tan\phi_1\tan\phi_2=\frac{a}{b};\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\therefore\ d\phi_1\sin 2\phi_2+d\phi_2\sin 2\phi_1=0.\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

Also, since HK, CI are equally inclined to CA, if an infinitesimal arc ds_1 is described by P through the motion of the trammel,

$$ds_1 = \pm IP\,d\phi_1, \qquad \text{(iii)}$$

according as the arc is measured from A or from B.



Again, since H, I, C, F are concyclic,

$$\angle FCH = \angle HIG = \phi_2;$$

$$\therefore \angle ICF = \phi_1 + \phi_2.$$

$$\frac{a}{\cos\phi_1\cos\phi_2} = \frac{b}{\sin\phi_1\sin\phi_2} = \frac{CI}{\cos(\phi_1-\phi_2)} = \frac{Ci}{\cos(\phi_1+\phi_2)}$$

$$= \frac{Ii}{\sin 2\phi_1} \text{ from } \triangle ICi;$$

$$\therefore IP = \frac{a+b}{2\cos(\phi_1-\phi_2)}\sin 2\phi_1. \qquad \text{(iv)}$$

Hence if a second figure were drawn with eccentric angle ϕ_2, the angles ϕ_1 and ϕ_2 having been interchanged and accented letters used in it, if s_2 denote the arc BP',

$$ds_2 = -I'P'd\phi_2 = -\frac{a+b}{2\cos(\phi_2-\phi_1)}\sin 2\phi_2;$$

$$\therefore ds_1 - ds_2 = \frac{a+b}{2\cos(\phi_1-\phi_2)}(\sin 2\phi_1 d\phi_1 + \sin 2\phi_2 d\phi_2)$$

$$= \frac{a+b}{2\cos(\phi_1-\phi_2)}(\sin 2\phi_1 + \sin 2\phi_2)(d\phi_1 + d\phi_2), \text{ by (ii)},$$

$$= (a+b)\sin(\phi_1+\phi_2)d(\phi_1+\phi_2)$$

$$= -d(CF);$$

$$\therefore s_2 - s_1 = CF,$$

no constant being required, since s_1, s_2 and CF vanish together.

E. M. Langley.

564. [D. 6. b.] "*A Trigonometrical Lucubration.*"

Under this heading in the May issue (pp. 66-68), Mr. Muirhead challenges readers to produce an elementary proof that if the inequalities

$$x>y>0, \quad \alpha>\beta>0$$

hold, and if the angles concerned are all in the first quadrant, then

$$\sin x\alpha/\sin y\alpha<\sin x\beta/\sin y\beta.$$

A simple proof could hardly be required to dispense with the further condition that the ratio of x to y is commensurable, which is implicitly used in Mr. Muirhead's own demonstration; this added restriction is necessary in what follows.

For all values of θ, ϕ, and k,

$$\sin(k+1)\phi \sin k\theta-\sin(k+1)\theta \sin k\phi$$
$$=\sin k\theta \sin k\phi(\cos\phi-\cos\theta)+\cos k\theta \cos k\phi(\tan k\theta \sin\phi-\tan k\phi \sin\theta)$$

If $\theta>\phi$ and the angles involved are all in the first quadrant, the first term on the right-hand side is positive, and the second is a positive multiple of the difference

$$\tan k\theta \operatorname{cosec}\theta-\tan k\phi \operatorname{cosec}\phi,$$

which will be denoted by $T(k)$.

Since identically

$$\{\tan k\psi-\tan(k-1)\psi\}\operatorname{cosec}\psi=\sec k\psi \sec(k-1)\psi,$$

the condition $\theta>\phi$ implies $T(k)>T(k-1)$, whence it follows, because $T(0)$ is zero, that $T(k)$ is positive for all integral values of k under consideration.

Hence $\sin(k+1)\phi \sin k\theta-\sin(k+1)\theta \sin k\phi$ is positive, and $\sin k\theta/\sin k\phi$ is greater than $\sin(k+1)\theta/\sin(k+1)\phi$;

$$\therefore \sin k\theta/\sin k\phi>\sin(k+l)\theta/\sin(k+l)\phi$$

for any positive integral value of l for which $(k+l-1)\theta$ is less than $\frac{1}{2}\pi$. That is, $\sin m\theta/\sin n\theta$ is less than $\sin m\phi/\sin n\phi$ if m and n are integers of which m is the greater, and writing

$$m=xp, \quad n=yp, \quad \theta=\alpha/p, \quad \phi=\beta/p,$$

where p may have any positive value, rational or irrational, we have the inequality required. E. H. NEVILLE.

565. [D. 6. b.] The following proof of the theorem:

$$\text{If } \frac{\pi}{2}>\theta>\phi \text{ and } n>1, \text{ then } \frac{\sin n\theta}{\sin\theta}<\frac{\sin n\phi}{\sin\phi},$$

appears simpler than that given by Mr. Muirhead in the May number of the *Gazette*.

As shown by Mr. Muirhead, under the given restrictions

$$\frac{\cos n\theta}{\cos\theta}<\frac{\cos n\phi}{\cos\phi}.$$

Writing for brevity C_r and c_r for $\cos\frac{n\theta}{2^r}$ and $\cos\frac{\theta}{2^r}$, C_r' and c_r' for the corresponding expressions for ϕ, we have $C_r/c_r<C_r'/c_r'$ for all finite values of r, while

$$\underset{r\to\infty}{\mathrm{L}}\, C_r/c_r=\underset{r\to\infty}{\mathrm{L}}\, C_r'/c_r',$$

and therefore

$$\prod_1^\infty C_r/\prod_1^\infty c_r<\prod_1^\infty C_r'/\prod_1^\infty c_r',$$

i.e. using Euler's product,

$$\frac{\sin n\theta}{n\theta} \cdot \frac{\theta}{\sin\theta} < \frac{\sin n\phi}{n\phi} \cdot \frac{\phi}{\sin\phi},$$

i.e. $\dfrac{\sin n\theta}{\sin\theta} < \dfrac{\sin n\phi}{\sin\phi}$, as required.

I have for some years employed this method to prove that $\dfrac{\sin\theta}{\theta}$ decreases as $\theta \to \dfrac{\pi}{2}$, but had not previously applied it to the inequality in question.

University College, Southampton. E. L. WATKIN.

566. [K'. 2. c.] (i) The locus of the mid-points of rays drawn from the orthocentre to the circumcircle is a circle whose centre is the mid-point of the line joining the circum- and orthocentres, and whose radius is $\dfrac{R}{2}$. It is called the *Nine Point Circle.*

* If P be any point on the circumcircle, let p be the mid-point of HP, where H is the orthocentre; and let N be the mid-point of OH. Join OP and Np.

Then $Np = \frac{1}{2}OP = \dfrac{R}{2}$ (a const.).

Hence locus of p is a circle centre N and radius $\dfrac{R}{2}$.

Cor. I. Since $HX = XH_1$, this circle passes through X and the corresponding points on CA and AB.

Cor. II. If α is mid-point of AH, the circle passes through α and the corresponding points on BH and CH.

Cor. III. Since $OA' = \alpha H$ and is parallel to it, $NA' = N\alpha$; hence the circle passes through A', B', C'.

(ii) The Simson Lines of the ends of a circumdiameter meet at right angles on the Nine Point Circle.

Let TT' be the ends of the circumdiameter, and t, t' the mid-points of TH, $T'H$. Then t, t' lie on the Nine Point Circle: they are also the points in which the Simson Lines of T and T' meet TH, $T'H$ (that these S. L. are at right angles is proved as usual). Also tt' passes through N. Hence the Simson Lines are lines at right angles through the ends of a diameter of the Nine Point Circle, and therefore they meet on the Nine Point Circle.

J. FITZROY JONES.

567. [R. 4. c.] *Pillory*, i. p. 160.

The framework is obviously deformable. If the question was set as a catch, it was grossly unfair; if it is a blunder, then such blunder is entirely inexcusable and lends support to the profane clamour for the Education of Examiners.

It is interesting to notice that, if motion in a vertical plane only is permitted, and if the framework is gradually lowered under support, then there is a position of equilibrium in which the bars AB and DE are inclined at 30°.

X. Y.

568. [K'. 8. a.] *Note* 551, p. 144.

In a paper on "The Brocard and Tucker Circles of a Cyclic Quadrilateral" (*Proceedings of the Edinburgh Mathematical Society*, vol. xxxvi. part 2) I have established the neater formula for ω, viz.

$$\operatorname{cosec}^2\omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B$$

on page 73 (15*b*), quoted by Mr. Genese.

As my paper was written after the note in vol. ix. of the *Gazette*, pp. 83-85, appeared, to which Mr. Genese refers, the above formula did not appear in the *Gazette*. FREDERICK G. W. BROWN.

* The reader is requested to draw the figure.

REVIEWS.

Matrices and Determinoids. By C. E. Cullis. Vol. I. pp. xii + 430, 21s., 1913; Vol. II. pp. xxiv + 555, 42s., 1918. (Camb. Univ. Press.)

The reviewer has to solve a difficult problem. On the one hand, readers have a right to some advice as to beginning the study of a work of which the end is by no means in sight, and as to embarking on the purchase of an unknown number of expensive volumes. On the other hand, not only was this exposition avowedly undertaken with a view to applications not yet disclosed, but Prof. Cullis's achievement will have to be estimated as a whole, for the damaging admission must be made that this elaborate and careful work does not convey an impression of *intrinsic* worth. To point out that we do not feel, after 950 pages of Hamilton or Grassmann, of Lie or Darboux, that we must suspend judgment for fear of being unjust, is to suggest a comparison that Prof. Cullis would doubtless be the last to claim, but the scale of the work leaves us no choice, for what we have to decide is whether the demands on time and energy, to say nothing of money, are reasonable.

The title, judged by the volumes before us, is somewhat misleading. Determinoids, other than determinants, play but a small part, nor has the author yet shown their invention to be of value. The determinoid of a matrix can be regarded—though this is not Prof. Cullis's point of view—as a combination of the largest determinants associated with the matrix, and it does not appear that the determinoid has any significant properties that do not belong to every homogeneous linear function of these determinants. And with regard to matrices and determinants, this is not a general survey of the immense field, but a meticulous examination of that small corner which is immediately inside the most obvious gate.

Generalisations of Laplace's development and Sylvester's identities, relations involving minors and their reciprocals and conjugates, the solution of linear algebraic equations, the properties of the rank of a matrix, and especially the connection between the rank of the product of a number of matrices and the ranks of the individual factors, are discussed exhaustively, within limits that will be inferred if it is said that elementary divisors and the characteristic equation have not yet found mention. In a reference library the volumes will be indispensable on account of their wealth of elementary formulae, but to the mathematician reading for pleasure, to the student desiring a broad education in the art, and to the worker in other fields looking for help, Prof. Bôcher's well-known little masterpiece is of incomparably greater value.

The arrangement of the material leaves something to be desired. For example, a glance at the index of Vol. II. shows the importance attached to the idea of a spacelet, but this idea is explained first in asides on pp. 78 and 208; to raise the spacelet to its proper dignity, "preliminary remarks, chiefly recapitulatory," occupy pp. 422 and 423, but by then the word has already occurred in large type and in many enunciations.

Repetition and detail, indefensible since the arguments are nowhere novel or subtle, make the work wearisome, and must have inflated the cost. Thus in § 3, Cases II. and IV. should have been dismissed in a sentence, § 4 could have been reduced to half a dozen lines, and in § 109 the whole of the second section, occupying *seven pages*, duplicates the first section but for the trivial interchange of rows with columns. An author appealing to readers who may be assumed to be highly qualified has no right to waste their time, and should mark with an asterisk those paragraphs that can profitably be omitted in every perusal.

It remains only to add that the volumes are produced with the skill always devoted at Cambridge to the big blue books. E. H. N.

Lectures on the Philosophy of Mathematics. By J. B. Shaw. Pp. viii + 206. 6s. net. 1918. (Open Court.)

Much has been written of late on the relations between Mathematics and General Philosophy, and the work of Poincaré and Russell is known to all.

The present author surveys the whole field of Pure Mathematics, and divides the subject into Static and Dynamic Mathematics. In the first division he places (1) Arithmetic, dealing with Integers, Rationals, the Continuum, General Ensembles and Functional Space; (2) Theory of Multiples, *i.e.* the various Geometries; (3) Tactic-comprising Arrangements and Combinatory Analysis; (4) Logistic and Mathematical Logic. The corresponding subdivisions of the Dynamic portion are (1) Theory of Operators, Groups, and Transformations; (2) Algebra, both ordinary and hypercomplex, finite and infinite; (3) Theory of Processes; (4) Theory of Inferences. In this classification a theory such as Function theory appears in several places under different aspects. The author then considers Mathematics as Theory of Form, of Invariancy, of Functionality and of Inversion, and finds that no one of these divisions can be considered to cover the whole field.

Throughout the book an interesting question is debated. Are mathematical theories discovered or invented? Are they free mental creations or do they exist in some ideal world, as Russell affirmed, before they are elaborated? The author takes the former view, and as neither belief seems susceptible of proof or disproof, he is obliged to employ the persuasive method. Considerations of history lead him to conclude that "mathematics is a creation of the mind, and is not due to the generalisation of experiences, or to their analysis; . . . and is the outcome of an evolution, the determining factors of which are the creative ability of the mind, and the environment in which it finds the problems which it has to solve" (p. 30). This is a natural enough assertion when we reflect on the historical progress of mathematics, or take the subject in its living and growing form. But if we consider a doctrine based on postulates, *e.g.* Group-theory, the other view that it is a discovery rather than an invention forces itself upon us; more especially when the subject matter can be approached in several quite different ways, as in the theory of algebraic functions and their integrals. It is then not surprising that Russell, with his anti-historical way of looking at things, should regard mathematics as a pre-existing realm awaiting the Columbus; and that Shaw, who looks on mathematics as an engine fashioned by man, should betray antagonism to Logistics. His hostility to this and the related work of Cantor leads him to affirm that "there is no such thing as the collection of all integers" (p. 78), which is a hard saying, for are there not many propositions concerning that collection? On p. 25 he tries to throw doubt on the proof that the real numbers form an unenumerable set. The proof, as he states it, involves Zermelo's axiom, which he does not mention, and which he seems to regard as obviously false. But the proof of Cantor, reproduced in Hobson's Real Variable, p. 70, involves no such axiom and is quite irrefragable.

Some other illustrations will show the point of view of our author. In dealing with quaternions, he states that the nature of the product of two vectors in space, *i.e.* whether it is vector, scalar, or what not, can never be determined from the vectors, but only from the (hyper-complex) numbers which the vectors represent and their character; while the usual view would be that vectors could be compelled to form "products" in any way desired, and the resulting calculus would be non-trivial if it could justify itself by its suggestiveness or utility; and of course there is no need for the vectors to represent anything. In dealing with numbers, he explicitly identifies the natural number 2, the signed number $+2$, and the ratio $2/1$, justifying this view by appeals to history and reasoning of a popular kind (p. 105 *et seq.*), which the reviewer, whose mind has been biassed by Peano, finds unconvincing. It is interesting, as well as surprising, to find a Mathematician with the width of knowledge of Prof. Shaw regarding the natural numbers as no better than the others. "God made the whole number, man made the rest," said Kronecker, but Shaw denies the celestial origin of them all.

The description of the various mathematical disciplines is sometimes sketchy, and the work is of a far more popular nature than Russell's books; the author being interested rather in the developments than in the foundation of mathematics. When talking round the subject, the author's style is more ornate than has been usual on this side of the Atlantic since the death of Sylvester.

The book is the outcome of a course of lectures delivered to a graduate club in Illinois University; and when we felt unkind, we reread the words

of the preface, "Critics are asked to keep in mind the purpose of the lectures." If that purpose is to provoke discussion, stimulate interest, and maintain enthusiasm, it has been fulfilled. A useful table illustrating the author's system of classification is given at the end of the book. H. G. F.

The Human worth of Rigorous Thinking. By C. J. Keyser. 7s. 6d. net. 1916. (Columbia University Press: Clarendon Press.)

Professor Keyser has collected in this book fifteen essays and addresses which were written at various times from 1900 onwards; ten of these deal with various mathematical subjects treated popularly, and five with educational matters.

The articles are of unequal value, and certain mannerisms of the author tend to exasperate the reader. He is fond of speaking, not in his own proper person, but of using phrases such as "our lecturer might say," "now, ladies and gentlemen, our lecturer will ask. . . ." This trick goes on through all the first address and much of the second. Repetitions frequently occur. Three times he quotes the "felicitous words" of W. B. Smith, "mathematics is the universal art apodictic," but nowhere does he explain the phrase; twice he mentions the request of the "brilliant author of 'East London Vision,'" to be informed of the human significance "of this majestic intellectual cosmos of yours, towering up like a million-lustred iceberg into the arctic night." In many places he elaborates his sentences and piles up his adjectives for effect rather than for clearness. The essay on the "Humanization of Teaching Mathematics," an address to schoolmasters, is a bad example of this. Nowhere does he give any practical hint as to how mathematics should be humanised. In fact, in much of the book, the matter is so thin that only the style can be criticised.

Other essays are on the question of the finitude of space, on four-dimensional space, and on the axiom of infinity. None of them is intended to be very deep. His criticism of the attempted proofs of the *existence* of infinite aggregate is, however, sound; and every attempt, he says "is destined before its birth to take the fatal figure of the wheel." He considers that no one who is not spiritually dead can contemplate an infinite sequence without emotion; and his views on the Aleph Cardinals would probably have surprised Cantor. "Whereas in former times the Infinite betrayed its presence not indeed to the faculties of Logic but only to the spiritual Imagination and Sensibility, mathematics has shown, even during the life of the elder men here present, . . . that the structure of Transfinite being is open to exploration by the organon of Thought."

In dealing with research at the American Universities his remarks are very candid and to the point. It seems that many professors and lecturers are so overburdened with elementary teaching and details of organisation that they have no time or energy left. There is an amusing instance of our author at his worst in the article on the graduate mathematical instruction of students who wish to know something of mathematics, but who do not intend to become mathematicians. Are there such students? he asks. "We have to do with what mathematicians call an existence Theorem. Do the students described exist? They do. Can the fact be demonstrated—deductively proved? It cannot. How then may we know it to be true? The answer is, partly by observation, partly by experience, partly by inference, and partly by being candid with ourselves."

The addresses and articles no doubt served a useful purpose when they were written, but they hardly deserve to be collected into book form.

H. G. F.

First Year Physics for Technical Schools. By G. W. Farmer. Pp. 183. 4s. 6d. 1920. (Longmans.)

An interesting booklet, enlivened by interesting little historical notes. Most of the argument is founded on the needs and happenings of daily life. It should prove stimulating to beginners, *e.g.* in continuation classes preliminary to courses at a Technical College; but whether it will help to develop the scientific habit of thought that is necessary for advanced work later on, when they reach a Technical College, is not very clear. J. M. C.

Mathematics for Collegiate Students of Agriculture and General Science. By A. M. Kenyon and W. V. Lovitt. Pp. 313. 1920. (The Macmillan Co.)

Hardly a work to be advocated for English schools. It is supposed to be a three-hours-a-week course for one year. A glance at the contents should be enough to prove the sanguine hopes of the compilers. Review of Equations, Graphical Representation, Logarithms, Trigonometry, Land Surveying, Statics, Small Errors, Conic Sections, Variation, Empirical Equations, Progressions, Annuities, Averages, Permutations and Combinations, Binomial Expansion, Laws of Heredity (Mendel), Compound Interest Law, Probability, Correlation. J. M. C.

School Statics. By W. G. Borchardt. Pp. 266. 6s. 1920. (Rivingtons.)

The volume forms Part I. of a *School Mechanics*, of which the general features are stated to be that the examples are mainly arithmetical, the bookwork simple, and easy experiments are suggested which can be performed either by the teacher or by the scholar. The order adopted is based on the experience of the author. Thus the Principle of the Lever comes first; and here it is good to see the definition of a moment given correctly as a "tendency to twist," and the product of the force and the arm defined as the "measure" of the turning moment. The extension of the idea to non-parallel forces is carefully developed. Then follow the Parallelogram and the Triangle of Forces, deduced from experiment; these are extended to forces in general, with a graphical section as sequel. There is a long and interesting section on machines, in which the Principle of Work is utilised. The book concludes with a chapter on easy frameworks, which are very clearly set forth; neat suggestions for laboratory experiments of a non-usual type; and a set of thirty-eight graduated test papers.

The volume should prove stimulating and successful. J. M. C.

Mensuration for Marine and Mechanical Engineers. By John W. Angles. Pp. xxviii, 160. 5s. net. 1919. (Longmans.)

This work is designed mainly for entrants to the 1st and 2nd Class Board of Trade Examinations for Marine Engineers; it seems to be admirably suited to that purpose. A special feature is the large number of worked-out examples, and another good point is that the numerous diagrams are clearly drawn and dimensioned. It will not be the fault of the author if students using this book fail to obtain a thorough knowledge of mensuration as applied to engineering, and to develop a capacity for reproducing that knowledge in a clear and well-illustrated manner.

An appendix is given in which the work involves integration; this is quite good, although the working out, probably in view of the type of student for which the work is intended, is rather prolix. It seems a pity that the methods of graphical integration are not given as alternatives. J. M. C.

An Elementary Treatise on Differential Equations and their Applications. By H. T. H. Piaggio. Pp. xvi+216+xxv. 12s. 1920. (Bell's Mathematical Series, Advanced Section.)

A book for undergraduates and their teachers. With a skill as admirable as it is rare, the author has appreciated in every part of the work the attainments and needs of the students for whom he writes, and the result is one of the best mathematical text-books in the language. The critic has nothing to do but to call attention to a few of the most valuable features and to one or two unlucky slips.

It is good to find the first chapter to be a geometrical discussion, all too brief, of a kind which in England has been associated to their common loss rather with the lecture-room than with the text-book.

In devoting an early chapter to some simple partial differential equations, Prof. Piaggio has put teachers and students alike under a debt which the latter can not realise. Is it credible that some of us became acquainted with the equations of wave motion and with their simpler solutions surreptitiously

in treatises on sound, because in pure mathematics a partial differential equation of the second order, however simple, was expected to yield precedence to the twenty-four solutions of the hypergeometric equation and to an abstract discrimination between general integrals, complete primitives, and the like ?

At only one point is the unaided reader likely to find the book unintelligible : at the beginning of Chapter V. he is invited to consider equations solvable for p *without having had a hint that* p *is ever an abbreviation for* dy/dx. To add to his confusion, the letter, which being consecrated in this subject to serve for dy/dx and for $\partial z/\partial x$ should not be otherwise employed, is one for which Prof. Piaggio has a natural but excessive fondness, and of the eleven uses which p has to serve in the book, four have already occurred before equations solvable for p are mentioned.

There is an excellent chapter on Frobenius' method of solution in series. The chapter that follows is concerned with existence theorems and shows the author as a brilliant teacher, but contains a curious flaw, the deduction from an inequality of the form

$$B_{n+1}/B_n < k_n/R + M/n, \quad k_n \to 1,$$

to the convergence of $\Sigma B_n x^n$ within $|x| = R$ being twice made by way of

$$\lim B_{n+1}/B_n = 1/R,$$

an assertion which is of course neither necessary to the conclusion nor implied by the premiss, though in fact it is in each case true in virtue of the actual definition of B_n; as if one should attempt a journey from Cambridge to London *via* both Hitchin and Bishop Stortford. Here it may be asked whether the view of x and y as *complex* variables in this connection only does not call for explanation, and even whether students presumably unacquainted with complex differentiation and integration should be encouraged to take as a matter of course the ordinary formal use of complex numbers in the solution of the linear equation with constant coefficients.

Other details may be criticised—for example, Fig. 13 must be seen to be believed—and small mistakes are to be regretted just because the book as a whole in scope and manner satisfies exactly a need widely recognised, and is to be recommended in the warmest terms possible. E. H. N.

The Laws of Mechanics. By S. H. Stelfox. Pp. 201. Price 6s. 1920. (Methuen & Co.)

This book does not pretend to be a text-book on Mechanics, but is a compilation of notes to be used in conjunction with a text-book by students who already have a working knowledge of the subject. Special stress is laid on graphical methods applied to dynamical problems, and sufficient explanation of the Calculus is given to enable the reader to understand the Calculus notation which is freely used.

Mr. Stelfox believes that much of the confusion in the use of units may be avoided by the adoption of dimensional arithmetic, an extension of the operations of arithmetic to the dimensional factor of a physical quantity, as illustrated in the following example :

A force of 3 lbs. acting on a given body is required to produce an acceleration of 8 ft./sec.^{2}; find the force necessary to produce an acceleration of 16 ft./sec.^{2} in the same body, and state the mass of the body :

$$M = \frac{F}{f} = \frac{3 \text{ lbs.}}{8 \text{ ft./sec.}^2} = \frac{3}{8}\,\frac{\text{lbs. sec.}^2}{\text{ft.}},$$

$$F = Mf = \frac{3}{8}\,\frac{\text{lbs. }\cancel{\text{sec.}^2}}{\cancel{\text{ft.}}} \times 16\,\frac{\cancel{\text{ft.}}}{\cancel{\text{sec.}^2}} = 6 \text{ lbs.}$$

After an explanation of the methods of compounding vectors, several illustrations are given of their application in the drawing of velocity and acceleration diagrams for a mechanism composed of rigid members.

Conditions of equilibrium are deduced from the principle that when an ideal machine is balanced under the forces acting on it the total power exerted is zero.

Examples are worked out showing how velocity diagrams may be drawn for any motion of a system of bodies, and from these any force acting on the system may be determined. When the motion is accelerated, a similar method is adopted by reversing the mass accelerations and considering the system as being in equilibrium.

Several applications of graphical methods to engineering problems are worked out in full in the last chapter: here the diagrams have been so much reduced that they are of little help, and the student will have to draw them for himself if he is to follow the text—doubtless this is what the author desired.

No examples beyond those worked out in the text are given to enable the student to test his knowledge of the principles that are expounded, but many engineers will welcome a book which gives in detail a graphical method of attacking such problems as the determination of the forces acting on a steam engine when the inertia of the reciprocating parts and the connecting rod is taken into account. R. C. FAWDRY.

Recreations in Mathematics. By H. E. LICKS. Pp. 155. 1.25$. 1917. (Van Nostrand Co.; Messrs. Constable.)

This volume is intended for the amusement of the man in the street, and for the young student who has a budding interest in matters mathematical. It is at times unconsciously amusing. We all know the story of the tumblers, one half full of wine and the other half full of water. A teaspoonful of wine is taken and placed in the other vessel, and so on and so on. Is the quantity of wine removed from the first greater or less than the quantity of water removed from the second? "Ball in his *Mathematical Recreations and Essays* says that the majority of people will say the former is greater, but this is not the case. H. E. Licks, who has studied this problem, claims that the two quantities are exactly equal." Again, to show that any number a is equal to any number b, we have the equation $(x-a)^3=(x-b)^3$. Taking the cube root of each gives the required result. With an air of conscious superiority, the author adds: "Perhaps the advanced student may be able to show that the equation has three roots, $x=\frac{1}{2}(a+b)\pm\frac{1}{6}\sqrt{-3}(a+b)$ and $x=\frac{1}{2}(a+b)$." The theory of logarithms "belongs in" Algebra. The astronomer who helped Julius Caesar to improve the calendar was Sioseneges. A good mnemonic for twelve places of π is:

"See I have a rhyme assisting,
My feeble brain its tasks sometimes as resisting"

But we fear that the exigencies of the rhyme have been admitted at the cost of the last three places. "Sin $B=b/a$ and sin $C=c/a$ in a right-angled triangle. Now it has been questioned by H. E. Licks whether this is the best way to define the sine for the beginner." Again "paradoxes and fallacies decrease as we ascend the mathematical ladder. . . . In *Analytic Geometry* H. E. Licks tried hard to find a fallacy, but without success." "Hundreds of problems similar to the above may be found in books and mathematical journals, hence H. E. Licks gives one not found in books, namely, to determine the path of a ray of light from a source S to the eye at E, when a transparent glass plate is interposed between." The following will be new to most of us: "John Phoenix, the first real humourist of America, was a graduate of West Point, and hence well versed in mathematics. In his essay called 'Report of a Scientific Lecture,' he alludes to the importance of adding a constant to the result of an integration. He says:

'By a beautiful application of the differential theory the singular fact is demonstrated, that all integrals assume the forms of the atoms of which they are composed, with, however, in every case the important addition of a *constant*, which, like the tail of a tadpole, may be dropped on certain occasions when it becomes troublesome. Hence, it will evidently follow that space is round, though, viewing it from various positions, the presence of the cumbersome addendum may slightly modify the definity of the rotundity. To ascertain and fix the conditions under which, in the definite consideration of the indefinite immensity, the infinitesimal incertitudes, which, homogeneously aggregated, compose the idea of space, admit of the computible retention of

this constant, would perform a beautiful and healthy recreation for the inquiring mind; but, pertaining more properly to the metaphysician than to the ethical student, it cannot enter into the present discussion.'"

After quoting 2 Kings xx. 11, Mr. H. E. Licks remarks: "On a properly constructed sun-dial, such as is described below, the shadow cannot go backward." As far as the spelling of names of months is concerned, the inventor J. W. Nystrom (? Mystrom), of "tonal numeration," is badly treated either by H. E. Licks or by De Morgan. The author pleads for indulgence if the arrangement of topics is incoherent. As he says, "some are long, some short, some are frivolous, some serious, some logical, and some absurd." There is plenty of interesting stuff amid all the incoherence, and there are very few actual slips or mistakes.

Lectures on Ten British Physicists of the Nineteenth Century. By A. MACFARLANE, pp. 114. 5s. 6d. net. 1919. (John Wiley, Chapman & Hall.)

(*Continued.*)

Croker replies:

"Mr. Babbage's invention is at first sight incredible, but if you will recollect those little numeral locks which one has seen in France, in which a series of numbers are written on a succession of wheels, you will have some idea of the principles of this machine, which is very curious and ingenious, and which not only will calculate all regular series, but also arrange the types for printing all the figures. At present, indeed, it is a matter of curiosity rather than of use, and I believe some good judges doubt whether it can ever be of any. But when I consider what has been already done by what were called Napier's bones and Gunter's scale, and the infinite and undiscovered variety of what may be called the *mechanical power* of numbers, I cannot but admit the possibility, nay, the probability, that important consequences may ultimately be derived from Mr. Babbage's principle. As to Mr. Gilbert's proposition of having a new machine constructed, I am rather inclined (with deference to his very superior judgment in such matters) to doubt whether that would be the most useful application of public money towards this object at present.

"I apprehend that Mr. Babbage's present machine, which however I have not seen, answers the purposes for which it was intended sufficiently well, and I rather think that a sum of money given to Mr. Babbage to reward his ingenuity, encourage his zeal, and repay his expenses, would tend eventually to the perfection of his machine. It was proposed at the Board of Longitude to give him 500*l*. out of the sum placed at our disposal for the reward of inventions tending to facilitate the ascertaining the Longitude. But the Board doubted that the invention was likely to be practically useful to a degree to justify a grant of this nature.

"I think you can have no difficulty in referring the matter to the Council of the Royal Society (which, although unworthy, I have the honour to be on), which by the assistance of its scientific members will give you the best opinion as to the value of the invention, and when that is obtained, it may be considered whether another machine should be made at the public expense or whether Mr. Babbage should receive a reward from Parliament or the Board of Longitude. . ."

It is also interesting to remember that the Italian savants took a great interest in the engines, and that General Menabrea published an account of the Analytical Engine in French.

And now *place aux dames*! This memoir was translated with notes into English by no less a person than Augusta Ada, daughter of Byron, "sole daughter of my house and heart." She was married to Lord Lovelace in the house in which Fielding used to live. Broughton, who was best man at Byron's wedding, says that the name Ada was from someone who had married into the family in the reign of King John. She had not inherited from her father his poetic gifts; indeed she was singularly lacking in appreciation of poetry, her mother on one occasion lamenting that Ada considered a then popular song to be quite as fine as any of her father's poems. Of her mother's capacity John Murray had a very high opinion, writing to the Ettrick Shepherd on April 10, 1815: "Could you not write a poetical epistle, a lively one, to Lady Byron?—she is a good mathematician, writes poetry, understands French, Italian, Latin and Greek." The mathematical capacity was the occasion of Byron's malignant sneers in his description of the "poor dear Mussulwomen" (*Beppo*, c. lxxviii.):

"They stare not on the stars from out their attics,
Nor deal, thank God for that, in mathematics."

Peter King, who defended William Whiston on a charge of heresy, was afterwards Baron King, and Lord Chancellor. The seventh baron wrote a Life of John Locke (the first Baron's maternal uncle was a Locke), and his second son was Peter John Locke King, a well-known M.P. and legal reformer. The

eighth baron was made Earl Lovelace. Lady Lovelace was suspected of being the author of *Vestiges of Creation*; she devoted most of her time to science and mechanical invention. Her notes to the memoir of Menabrea were her own, and the algebraical working of the various problems was also hers, except, says Babbage, "that relating to the numbers of Bernoulli, which I had offered to do" to save her the trouble. This she sent back for amendment, having "detected a grave mistake which I had made in the process." Her notes "extend to about three times the length of the original memoir," so that her share in the work was by no means merely perfunctory, since "she entered fully into almost all the very difficult and abstract questions connected with the subject." She brought up her eldest son to be a sailor, and her second to be a civil engineer. Originality was also marked in her daughter the Baroness Wentworth, who died only four years ago. At twelve she exhibited an architectural drawing at the Royal Academy. She was a musician, mathematician, and linguist, and her chess play was of a very high order. She accompanied her husband, Wilfred Blunt, in his Arabian and Egyptian wanderings, and at the age of 80 published her great book on Arabian horses, of which she had studs in Egypt and at Crabbet Park, Sussex (the house at which she herself designed).

Babbage's *Decline of Science in England* deserves to be read and remembered, not merely because history repeats itself in the relations between the "powers that be" and the exponents of science in this country, but because the British Association came into existence from the discussion that followed its publication. The Royal Society of those days had lost much of its influence, as may be seen from a study of Faraday's letters, of De Morgan's *Budget*, and the like. Years before this, the formation of subsidiary societies—such as the Linnaean—was condemned as a contributory cause of this decay by the President, Sir Joseph Banks, in the graphic phrases: "They will go on docking the skirts of the old lady's petticoats till she has not a rag left to cover her nakedness." But the cause was more deeply rooted than the courtier Banks could see. The breed of men who almost broke the stout heart of a Babbage is with us still—the men who, as De Morgan said, do not care a twopenny (Indian coin) for all the science in the country, men like the general who cried: "What do we want with science, we want to win the war!" Yes! they are still with us, and as Talleyrand said of the Bourbons: "*Ils n'ont rien appris, ni rien oublié*," or as Russian Alexander put it: "*Ils sont incorrigés et incorrigibles.*" Even Sir David Brewster, whose services to science had received such measure of reward as a K.H. implied, after reviewing *The Decline* in the *Quarterly*, significantly closed his article with the words of the Wise Man: "I returned and saw under the sun that there is neither yet bread to the wise nor yet riches to men of understanding, nor yet favour to men of skill." An attempt was made to answer Babbage's damning indictment. It bore the attractive title: "On the Alleged Decline of Science in England by a Foreigner," and had an introduction by Faraday. The alien was one Gerard Mohl, Professor of Astronomy and Mathematics at Utrecht. But there was too much truth in the charges, and Babbage knew his ground too well to make an "answer" effective. For it must not be forgotten that Babbage had a reputation for careful statement which made him an inconvenient antagonist to those who were aware of his powers. "There was no branch of Science," says Sir F. Pollock, "to which Babbage did not make some valuable addition, or upon which he did not throw some light. But for his engrossment by the calculating engines and all the troubles and annoyances to which they gave rise, Babbage would probably have made great discoveries, and would have been eminent as a physicist. Now his name is chiefly known as that of an inventor struggling with the government of the day for recognition and reward, or rather for payment and assistance, and for his squabbles with other men of Science. He was very generous to the Swedish inventors of a calculating machine, founded on his own, but with practical improvements, and assisted them in every way." It was said of him that probably he never penned a sentence in which lurked the least obscurity, confusion, or contradiction of thought. There have been few men of whom that could be said. It has been stated that everything he did was fragmentary. In taking leave of Babbage we might point out that he was the delight of all London

for his unbridled hatred of Organ-grinders, whom he pursued with a whole-hearted hatred which had nothing fragmentary about it.

His last surviving son, Major-General Henry Prevost Babbage, died a few months ago at the age of 94. He retired from the army and lived at Cheltenham, and there, in his own workshop, he constructed after he was 80 years of age a calculating machine for the use of astronomers.

The next of the physicists is Whewell.

The Lion and the Eagle of Trinity were joined by the Peacock of the College, *i.e.* in less parabolic language, Whewell, Sedgwick and Peacock, the Dean of Ely, found themselves together. Sedgwick, being challenged as to his thoughts, replied: "I was thinking what a strange chance it was that brought together the three ugliest men in Cambridge." Peacock, who we believe was by general consent the ugliest of the three, answered: "Speak for yourself, Sedgwick."

For his lecture on "The Lion of Trinity" Macfarlane had the letters in the *Life* published by Whewell's niece, Mrs. Stair Douglas, and in one of the two volumes of the Todhunter *Writings*. The most graphic description of the rugged man that we can remember is by one who is still alive, Mr. T. G. Bonney: To most persons that broad forehead with its massive brow seemed indicative of intellectual strength, almost gigantic; the square shoulders, strong bones and muscles, the swinging gait—with which, as he swept along he seemed to shoulder aside weaker men by the very waft of his passing—told of irresistible force of will and energy of purpose; tenderness of heart seemed improbable in one of such Titanic mould; one deemed him a 'man of iron,' who, had he chosen a field other than literature and science, might also have been one of blood; but, as we shall presently see, underneath that rough exterior there was a warm and affectionate heart concealed."

"There was once a Master of Trinity whose name was easier to whistle than to pronounce. Whewell's nickname was Billy Whistle, and it is said that Whewell, hearing a row after midnight, burst open his own door, which had been firmly tied up by rope, and made for a lonely figure standing by a piano in the middle of the lawn. He chased the offender thrice round the colonnades of Nevile's Court, caught him, and asked, panting: "Do you know who I am, Sir?" The answer came promptly: "Yes! Old Whistle who made that mistake in his *Dynamics*." This was perhaps understating the facts, for not merely did the volume of 1823 contain a page of Errata, but 7 pp. of *Corrections* were afterwards issued. The name of "Lion" may have been first used of Whewell by Tennyson, who used to write of him as the "lion-like man." The story goes that Whewell overlooked Tennyson's inattention at lectures. For instance, when he caught him reading a Virgil under his desk, he interrupted his lecture by asking: "Mr. Tennyson, what is the Compound Interest of a penny put out at the Christian era up to the present time"?

Whewell went up to Trinity in 1812. We have seen what Babbage had to pay for books. In 1814 Whewell had to pay ten pounds for a copy of Newton's *Principia*, "a very scarce book in three quarto volumes, most elegantly bound." In that year, to the consternation of his old schoolmaster, who disapproved of his promising pupil spending his time at anything but mathematics, he won the Chancellor's Medal for a Prize Poem on 'Boadicea.' It may have had some effect on his place in the Tripos, but he lamented to his father that he had not paid sufficient attention to the art of writing rapidly, a complaint which reminds us of Parkinson's year. At any rate he fulfilled the prediction of his future College Tutor, who some nine years before had prophesied that he would be among the first six Wranglers. Whewell and Edward Jacob of Caius (then under twenty, whereas Whewell was twenty-two) were far ahead of all rivals, and the next eight on the list were bracketed, Sheepshanks being the only man whose name became afterwards of note in the scientific world. Edward Jacob, the Senior, and also first Smith's prizeman, was the son of a noted writer on Political Economy. He went to the bar, but his life was cut short before he had time to win any of the greater prizes of an exacting profession. It was said that Jacob's accuracy was far greater than Whewell's, that Whewell nevertheless felt inclined to "challenge"

his successful rival, but was assured by the moderator that he could not defeat Jacob even if he were to write all day and if Jacob wrote nothing. But the story is not of Whewell and his supplanter, Jacob. When Pollock was Senior in 1806, Hornbuckle of John's, the moderator, congratulated him. "But I may be challenged," said Pollock. To which the reply was: "You may sit down and do nothing, and no one would get up to you in a whole day."

In due course Whewell became a Tutor of Trinity, and the following anecdote, if true, gives an illustration of the old views on the relations of the College officials to the undergraduates. It was the custom of the Tutor to give from time to time a wine-party after Hall, and the servant to whom Whewell gave the list of men to be invited noticed that it included a young undergraduate who had died some weeks before. He pointed out the fact to the Tutor, who sternly reprimanded him: "You ought always to tell me when one of my pupils dies." As a mathematician he was not considered by his contemporaries as really of the highest rank, but this was probably because his bent was analytical rather than geometrical. Pollock, the Senior of 1806, wrote to De Morgan: "Whewell was a very, *very* considerable man, but not, I think, a great man." As a "coach" he might have rivalled his great successors, for among his first pupils was King, the Senior of 1819, afterwards President of Queens' and Lucasian Professor. So distinctly was King's superiority marked that it was said that the gap between the first and second on the list was greater than ever known before. Another pupil, who was expected to be high among the Wranglers, but got only a second class, was the brother of Lord Brougham. Whewell had the reputation of not doing his best for his pupils.

On Whewell's wedding day—the letter reached him on his honeymoon at Coniston—Christopher Wordsworth wrote to inform him that he was resigning the Mastership of Trinity. The unhappy bridegroom was torn from his "hymeneal Elysium," had to leave his bride and post off to Sir Robert Peel. The sequence of events was—Married Oct. 12th, accepted Mastership Oct. 18th, saw Peel Oct, 19th, and back to Coniston by the night mail. Little objection was expressed to the appointment. The situation was expressed by Martin, who refused to stand against Whewell, in: "Nobody could supply his place as a Fellowship Examiner either in the Morals and Metaphysics or in the Mathematics." Of course opinions varied with regard to him both as man and as official. The undergraduate, especially if rather older than the average, was often driven to regard the great Master as a fussy martinet, as incredibly punctilious about trifles. Venables found satisfaction in the fact that Whewell's humbug and imbecility limited each other. On the other hand, a private pupil might, like Kenelm Digby (author of the *Broadstone of Honour*), become a life-long friend, and write in terms like these:

> "Whewell was one of the most generous, open-hearted, disinterested and noble-minded men I ever knew. I remember circumstances which called for the exercise of each of those rare qualities, when they were met in a way which would now seem incredible, so fast does the world seem moving from all ancient standards of goodness and moral grandeur."

The anonymous *Plurality of Worlds* was written in answer to Chalmers, who, in his *Astronomical Discourses* had maintained the view that there are more worlds than one. Todhunter quotes from Coleridge's "flippant remark," which is almost a solitary expression of a contrary opinion:

> *A lady once asked me:* "What then could be the intention in creating so many great bodies, so apparently useless to us? I said I did not know except to make dirt cheap."

The essay gave rise to many epigrams, of which the best was by Sir Francis Doyle. Here is one of the many variants:

> "Though you through the regions of space should have travelled,
> And of nebular films the remotest unravelled,
> You'll find, though you tread on the bounds of infinity,
> That God's greatest work is the Master of Trinity."

I forgot who it was who remarked that Whewell thinks himself a fraction of the universe, and wishes to make the denominator as large as possible; or again, that Whewell had treated the inferior planets with a graduated

scorn, nicely proportioned to their distance from the Lodge at Trinity. Sydney Smith's famous "science is his forte and omniscience is his foible" is quoted by Mr. Macfarlane in this lecture.

Whewell was always ready to enrich the ante-chapel of Trinity with stained glass, busts, statues, etc. He successfully opposed the suggestion to place in the ante-chapel Thorwaldsen's statue of Byron, which at one time was derelict in the Custom House vaults in London, after being refused admission to Westminster Abbey. His suggestion that the Trinity Library would be a suitable place was eventually adopted. I have before me a letter showing how closely he watched over the interests of the Society in such matters.

"TRIN. LODGE, CAMBRIDGE,
Apr.l 5, 1848.

MY DEAR CHIEF BARON.

When you were here I believe I mentioned to you that we have in the stained glass windows of the College Hall the arms of most of our great lawyers who have been of the College, and that we are about to add some new ones. It would be a great gratification to the College if you would, on this occasion, give us your arms to take their place with the others. The work is to be executed by Willement.*

Believe me, my dear Chief Baron,
Yours very truly,
W. WHEWELL.

SIR FR. POLLOCK, &C."

He gave the statue of Bacon, he wrote the inscription for the statue of Sheepshanks, by Foley, and when Lord Lansdowne offered a memorial statue, Macaulay supported the claims of Bentley, while the Master and Fellows were in favour of Barrow. Airy notes, in 1858, that Cromwell (a member of Sidney Sussex) was not suggested. Sedgwick's remarks are worth quoting :

"Barrow was a great inventive mathematician, and a forerunner clearing the way for the vast discoveries of Newton. He was great also as a moralist and divine, and was a man of gigantic learning. What grand old-fashioned eloquence, what earnestness in the cause of moral truth, do we find in the works of Barrow! They will live so long as majestic eloquence and moral truth and Christian hopes are dear to the hearts of men."

De Morgan wrote to his old teacher of more than forty years before :

"The Master of Trinity was conspicuous as a rough customer, an intellectual bully, an overbearing disputant. The character was as well established as that of Sam Johnson, but there was a marked difference. It was said of Johnson that if his pistol missed fire he would knock you down with the butt end of it; but Whewell, in like case, always acknowledged the miss, and loaded again or not, as the case might be."

The reference to Johnson is irresistible, and oddly enough there is another suggested in the name of Lady Affleck, the widowed sister of Leslie Ellis, who became Whewell's second wife. Affleck is the old pronunciation of Auchinleck, Boswell's ancestral seat.

Whewell was the author of some amusing sayings, and the following may have been unconsciously the source of the later T. and T′. Airy found that he required more help at the Cambridge Observatory than his first assistant, Balder, could give him. Some discussion arose as to what the title of the new observer should be, and Whewell said : "Of course he must be called Balder-dash."

He was killed by a fall from his horse. His brain was found to weigh below the average—49 oz. Dr. Humphrys said that he would have had a stroke in less than 5 months.

Much of the "imperious combativeness" of the "Professor of Casuistry" left him as the years rolled on. His labours for his College and for the University were of priceless value to both, and in the renovation of the Baconian philosophy, as Macfarlane says, he has left an enduring monument. Perhaps the best of the many tributes paid to his memory is that by Tom Taylor in *Punch*, of March 17, 1866. Todhunter's closing words are worthy of reproduction :

"One who was thoroughly acquainted with him, and whose own long career has been conspicuously bright and honourable, when lately recalling the incidents of that career, said that he had known many famous men, but on the whole Dr. Whewell was the foremost of them all."

(*To be continued.*)

* The name at end of Whewell's letter is difficult to decipher. Mr. W. W. Rouse Ball has kindly identified it as Willement (1786-1871).

NOTICE.

The provision of rooms where readers may inspect Scientific works before deciding upon purchase is a welcome innovation. We have received notice that such rooms have been provided by Messrs. Ginn & Co., 7 Queen Square, Southampton Row, and Messrs. Sotheran & Co., at 140 Strand, W.C. 2.

THE PILLORY.

(i) A rod of length $2a \sin \alpha$ rests in equilibrium in a rough hemispherical bowl of radius a, being in a vertical diametral plane with its upper end just below the horizontal rim of the bowl. Prove that the coefficient of friction is

$$\operatorname{cosec} 2\alpha\,\{(1+4\cos^2\alpha)^{\frac{1}{2}}-1\},$$

and that there is no lower position of limiting equilibrium in a vertical diametral plane.

(ii) If I_1, I_2, I_3 are the centres of the escribed circles of the $\triangle ABC$, and if r, ρ are the radii of the circles inscribed in ABC and I_1, I_2, I_3 respectively, prove that

$$r\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)=\rho\left(\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}\right).$$

Magdalen, Brasenose, Christchurch, Worcester, March 1919.

A. B. Mayne.

ERRATUM.

Reader, Carthagena was of the mind that unto those three things which the Ancients held impossible, there should be added this fourth, to find a book printed without Erratas. It seems that the hands of Briareus and the eyes of Argus will not prevent them.—Cotton Mather's *Magnalia Christi Americana*, 1702.

P. 136 (the word before the figure), *for CA read AB.*

ADDENDUM.

An inexplicable omission has been pointed out by several readers. Neither the "Teacher's Library" (*Gazette*, Dec. 1919) nor the list on p. 409 in the same number contains: *A Study of Mathematical Education*, by Mr. Benchara Branford. 5s. net. 1908. (Oxford Univ. Press.)

THE LIBRARY.

Change of Address.

The Library is now at 9 Brunswick Square, W.C., the new premises of the Teachers' Guild.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

Please note Librarian's change of address:—C. E. Williams, M.A., 30 Carlton Hill, St. John's Wood, London, N.W. 8.

GLASGOW: PRINTED AT THE UNIVERSITY PRESS BY ROBERT MACLEHOSE AND CO. LTD.

BOOKS RECEIVED, CONTENTS OF JOURNALS, ETC.

December, 1920.

The Purpose of Zeno's Arguments on Motion. By F. CAJORI. Pp. 7-20. (Reprint from *Isis*, Jan. 1920.)

Géométrie synthétique des Unicursales de troisième classe et de quatrième ordre. Pp. vi+97. n.p. 1920. (Gauthier-Villars.)

The Dissection of Rectilineal Figures. (Reprinted from the *Messenger of Mathematics*, Feb. 1919 and November 1919.) By W. H. MACAULAY.

A First Course in Nomography. By S. BRODETSKY. Pp. x+135. 10s. net. 1920. (Bell & Sons.)

A History of the Conceptions of Limits and Fluxions in Great Britain, from Newton to Woodhouse. By F. CAJORI. Pp. viii+299. 8s. 6d. net. 1920. (Open Court Co.)

Wiskundige Opgaven met de Oplossingen. Dertiende Deel. 2de Stuk. 1919. (H. C. Delsman, Amsterdam.)

Table de Caractéristiques de base 30030, *donnant en un seul coup d'œil des facteurs premiers des nombres premiers avec* 30030 *et inférieurs à* 901800900. By E. LEBON. Tome I. Fascicule I. Tableau $I.\,I'=B_k+1$.

Table des Caractéristiques $K<30030$, *K variant de* 1 *à* 4680. By E. LEBON. Pp. xx+56. 15 fr. (majoration temporaire 100%). 1920. (Gauthier-Villars.)

William Done Bushell, of Harrow. By his Son, and others. Pp. 74. 1920. (Camb. Univ. Press.)

A Second Course in Mathematics for Technical Students. By F. J. HALER and A. H. STUART. Pp. viii+363.

The Laws of Mechanics, A Supplementary Text-Book. By S. H. STELFOX. Pp. ix+201. 6s. 1920. (Methuen.)

Everyman's Mathematics. By F. W. HARVEY. Pp. 138. 4s. 1920. (Methuen.)

A School Geometry. By B. H. HOWARD and J. A. BINGHAM. Pp. xxvi+370. 5s. 6d., or in two parts, 3s. 3d. each. 1920. (Hodder & Stoughton.)

A Geometry for Schools. By A. C. JONES. Parts I. pp. 96, 2s.; II. pp. 97-224, 2s. 6d., III. pp. 225-327. 2s. 6d. n.d. (E. Arnold.)

Mathematical Papers for Admission into the R.M.A. and R.M.C., and Papers in Elementary Engineering for Naval Cadetships and R.A.F. Nov. 1919 and July 1920. Edited by R. M. MILNE. Pp. 34. With Answers. 1920. (Macmillan.)

Some Famous Problems of the Theory of Numbers, and, in particular, Waring's Problem. An Inaugural Lecture delivered before the University of Oxford. By G. H. HARDY. Pp. 34. 1920. (Oxford, Clarendon Press.)

Bulletin de L'Institut Aérodynamique de Koutchino. Fasc. VI. Pp. 101. 1920. (Gauthier-Villars.)

The Elements of Plane Geometry. By C. DAVISON. Pp. 1-280. 10s. net. 1920. (Cam. Univ. Press.)

An Introduction to Combinatory Analysis. By P. A. MACMAHON. Pp. viii+71. 7s. 6d. net. 1920. (Cam. Univ. Press.)

Veicoli al Servizio dei Calcolatori. By GINO LORIA. Pp. 77-93. Reprinted from *Scientia*, Aug. 1920.

Exercises on Arithmetic. Arranged in Two Courses. By A. E. LAYNG. Pp. 230+31. 3s. 6d.; with answers, 4s. 1920. (John Murray.)

Saccheri's Euclides Vindicatus. Edited and translated by G. B. HALSTED. Pp. xiii+xxx+246. 10s. net. 1920. (Open Court Company.)

Vocational Mathematics. By W. H. DOOLEY, revised by A. RITCHIE-SCOTT. Pp. viii+311. 5s. net. 1920. (Heath & Co.)

Newton. By GINO LORIA. Pp. 69. 2 l. 1920. (Formiggini, Rome.)

The American Journal of Mathematics.

April, 1920.

On the Convergence of Certain Classes of Series of Functions. Pp. 77-90. R. D. Carmichael. *On the Solution of Linear Equations in Infinitely Many Variables by Successive Approximations.* Pp. 91-96. J. L. Walsh. *Self-Dual Plane Curves of the Fourth Order.* Pp. 97-118. L. E. Wear. *On the Groups of Isomorphisms of a System of Abelian Groups of Order p^m and Type* $(n, 1, 1, \ldots, 1)$. Pp. 119-128. L. C. Mathewson. *On the Satellite Line of the Cubic.* Pp. 129-135. R. M. Winger.

July, 1920.

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The American Mathematical Monthly.

June, 1920.

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Oct. 1920.

The Average Reading Vocabulary. An Application of Bayes's Theorem. Pp. 347-354. W. Weaver. *A Graphical Aid in the Study of Functions of a Complex Variable.* Pp. 354-357. N. Miller. *Modular.* Pp. 357-361. A. A. Bennett. *Discussions. Note on the Solution of Fractional Equations.* Pp. 366-368. C. A. Noble. *Determination of an Angle of a Right Triangle, without Tables.* Pp. 368-369. R. A. Johnson.

Annals of Mathematics.

June, 1920.

Algebraic surfaces, their Cycles and Integrals. Pp. 225-258. S. Lefschetz. *The Potential of Ring-shaped Discs.* Pp. 259-275. E. P. Adams. *Total Differentiability. A Correction.* Pp. 276-277. E. J. Townsend. *Existence Theorem for the Non-self-adjoint Linear System of the Second Order.* Pp. 278-290. H. J. Ettlinger. *Motion in a Resisting Medium.* Pp. 291-298. J. K. Whittemore. *Continuous Matrices, Algebraic Correspondences, and Closure.* Pp. 299-305. A. A. Bennett. *Urn schemata as a basis for the Development of Correlation Theory.* Pp. 306-322. H. L. Rietz. *On Pseudo-resolvents of Linear Integral Equations in general Analysis.* Pp. 323→. T. H. Hildebrandt.

Sept. 1920.

On Multiform Functions defined by Differential Equations of the First Order. Pp. 1-10. P. Boutroux. *Hermitian Metrics.* Pp. 11-28. J. L. Coolidge. *On the Expansion of Certain Analytic Functions in Series.* Pp. 29-34. R. D. Carmichael. *Notes on the Cyclic Quadrilateral.* Pp. 35-42. F. V. Morley. *Notes on Preceding Paper.* Pp. 43. F. Morley. *Qualitative Properties of the Ballistic Trajectory.* Pp. 44→. T. H. Gronwall.

Bulletin of the American Mathematical Society.

June, 1920.

The Sum of the Face Angles of Certain Polyhedrons in n-Space. Pp. 398-399. H. F. MacNeish. *On Bordered Fredholm Determinants.* Pp. 400-405. T. H. Hildebrandt. *On the Coherence of Certain Systems in General Analysis.* Pp. 405-408. A. D. Pitcher. *On the Adjoint of a Certain Mixed Equation.* Pp. 408-412. R. F. Borden.

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L'Enseignement Mathématique.

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C. A. Laisant. Pp. 73-80. *Extension du problème des triangles héroniens.* Pp. 81-85. C. A. LAISANT. *Sur l'élimination algébrique.* II. Pp. 85-105. CH. RIQUIER. *Table de caractéristiques de base* 30030 *donnant en un seul coup d'œil les facteurs premiers des nombres premiers avec* 30030 *et inférieurs à* 901800900. Pp. 105-117. E. LEBON. *Sur la théorie des vecteurs ; essai de calcul symbolique.* Pp. 117-131. TH. ROUSSEAU. *Calcul des racines réelles d'une équation algébrique ou transcendente par approximations successives.* Pp. 131-135. MLADEN-T. BERITCH.

The Journal of the Indian Mathematical Society.

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